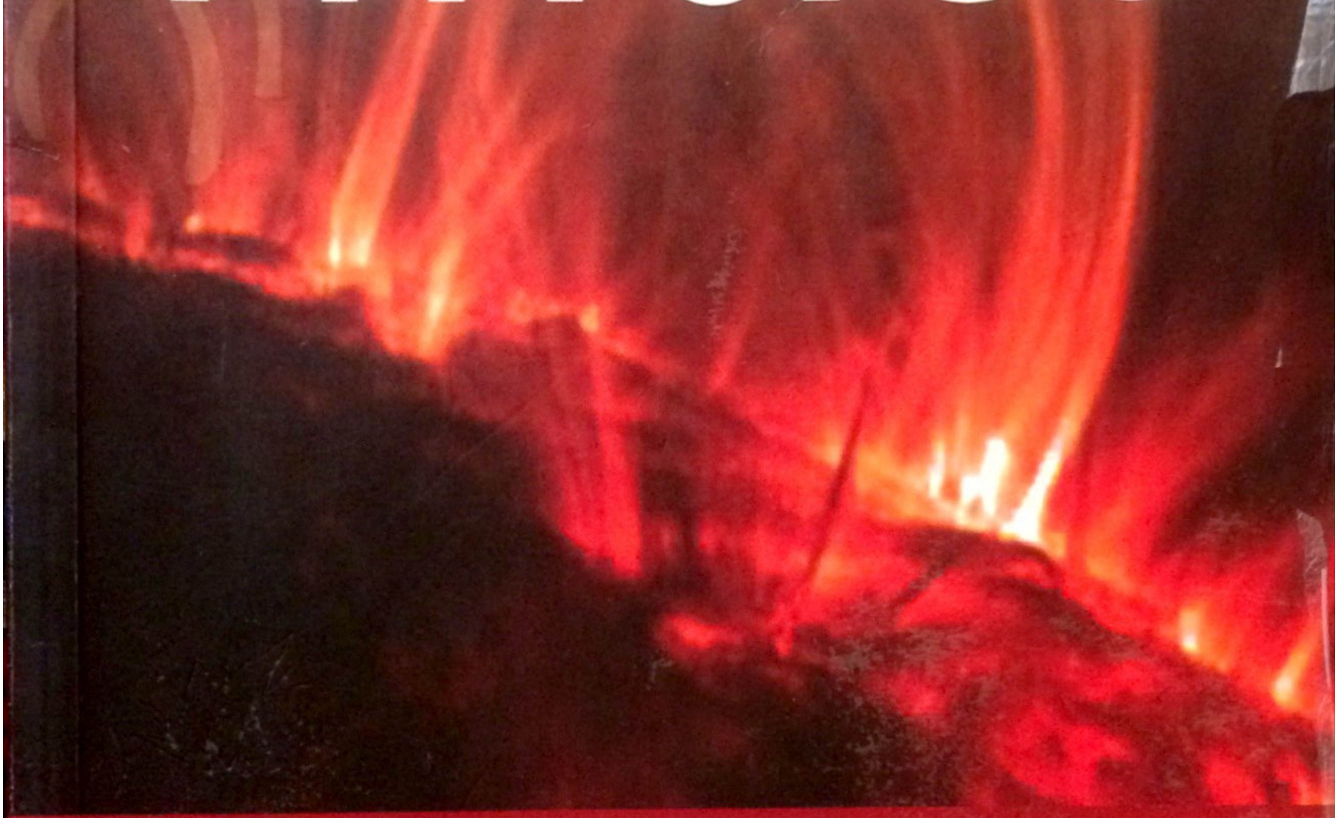
Textbook of PLASMA PHYSICS



Suresh Chandra

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PLASMA PHYSICS

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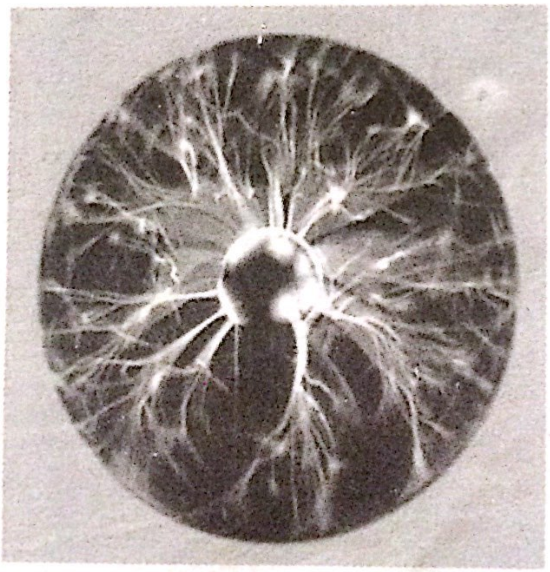
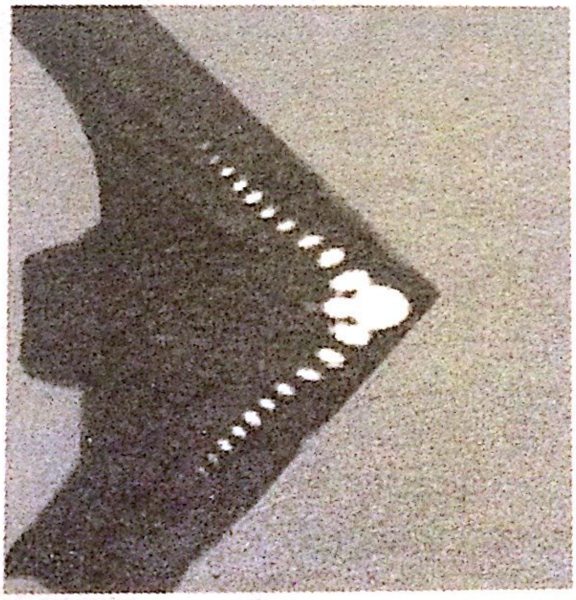
In pious memory of

my beloved sister

Ms Krishna Sharma

who left for heavenly abode
on 10 June 2008





Preface

Plasma is being considered as one of the viable sources to meet out future demand of energy. Research work in the field of plasma science is extensively going on all over the world. Plasma Physics is one of the courses being taught to postgraduate students in India as well as abroad. I happened to teach this course to MSc Physics students at DDU Gorakhpur University, Gorakhpur and SRTM University, Nanded (Maharashtra). While teaching the course, as a habit, I prepared my notes. On persistent demand from my students and valuable advices from my friends and colleagues, this book has been developed out of my notes, which have been revised from time to time.

While preparing the manuscript of the book, I have been helped, advised, and encouraged by my seniors, friends and colleagues working in various institutions/universities in India as well as abroad, and by my friends in personal life. I am heartily thankful to all of them. I would like to thank my students for their feedback for improvement of the notes. I would like to thank authors and publishers of those books which I consulted for my teaching as well as during preparation of this manuscript.

I am grateful to Prof. JV Narlikar, Prof. Dr WH Kegel and Prof. SP Khare for their encouragement. I am thankful to my PhD students, Mr Bhagwat K Kumthekar, Mr GM Dak and my son, Mohit K Sharma for their help in preparation of the manuscript and verification of some derivations. Special thanks are due to my wife Purnima Sharma for her valuable cooperation in my life and for sharing a major part of the responsibility of family affairs, so that I could spend my time for this book. In the last but not the least, I am highly thankful to the publisher for bringing out this book nicely and in a very short time.

Suresh Chandra

Contents

1
Plasma State

Three states of matter, known as solid, liquid and gas, are generally known to a common person. Any one of these states can, in general, be converted into another through the exchange of energy. In day-to-day life, H_2O molecule is a remarkable example, as it is found in all the three states, as ice (solid), water (liquid) and steam (gas). Obviously, by supplying energy to the matter, solid state can be converted into a liquid one, and a liquid state into a gaseous one; for the reverse process, the energy is extracted out from the matter. In some cases, it is possible to convert solid state into a gaseous state. NH_4Cl is example of such case. Further supply of energy to a matter in the gaseous state, breaks the molecules into its constituent atoms, and then the atoms are stripped off their electrons producing, positive ions and negatively charged electrons. The amount of energy required to liberate an electron from an atom is known as the *ionization potential*[1]. This supplied energy may be in the form of heat, radiation, due to collision or due to chemical reaction. Ionization due to heat occurs at very high temperature of the order of million Kelvin which can be produced in the laboratories. This state of matter where charged as well as neutral particles exist simultaneously is generally known as plasma. This term 'plasma' was first coined by Langmuir in 1923. Thus, the plasma may be defined as the following:

Plasma is a quasi-neutral gas of charged and neutral particles which exhibit collective behaviour.

As the plasma is found in a natural form in a number of cosmic objects and in the upper atmosphere of the earth, therefore, it is sometimes

[1]From an atom, having more than one electron, electrons may be liberated one by one. Thus, an atoms can have several ionization potentials; each subsequent ionization of an atom has larger value of ionization potential.

defined as

the fourth state of matter.

When the gas is ionized, its dynamical behaviour is influenced by the external electric and magnetic fields. Moreover, the separated charged particles within the plasma give rise to new forces between the constituent particles. Thus, the properties of the plasma become quite different from those of neutral atoms and molecules. In this chapter we shall discuss some processes and properties of plasma.

1.1 Natural plasma

Natural plasma exists in some cosmic objects like interiors and atmospheres of hot stars, planetary nebulae, regions of ionized hydrogen in the interstellar medium, and the upper atmosphere of the earth. On the earth, plasma can however be produced in the laboratories. The reason for finding natural plasma in the cosmic objects and not at the earth is due to the large differences in density and temperature in the two regions. On the earth, the density is very large and temperature is very low as compared to those in the cosmic objects. Some regions where natural plasma is found are as the following.

1.1.1 Ionosphere

Streams of charged particles, known as *solar wind*, are being continuously emitted by our sun. Some of these charged particles reach up to the upper atmosphere of the earth. Moreover, intense radiations (γ-rays, x-rays, UV radiations) coming from the outer space ionize the upper atmosphere of the earth. This upper atmosphere of the earth is known as the *ionosphere*. It is about 50 km above the earth's surface. Since hazardous radiations (γ-rays, x-rays, UV radiations) are absorbed in the upper atmosphere of the earth, the earth's atmosphere thus plays important role in our life by shielding us from the hazardous radiations coming from the outer space. This ionosphere is used for communication purposes, as the radiations of frequency less than the plasma frequency are reflected back by the plasma in the ionosphere .

1.1.2 van Allen belts

The investigations made by satellites found two regions, known as the *van Allen radiation belts*, which envelop the earth. One belt is at a distance of about 9700 km and second at 22,500 km from the surface of the earth. The thickness of inner and outer belts is respectively about 4800 km and 8000 km. The belts contain charged particles (ions) trapped between the magnetic lines of force.

1.1.3 Aurorae

Above the earth's magnetic poles, the charged particles have free access to the earth's surface, as the lines of magnetic field are concentrated towards the surface. These charged particles interact with the molecules of the upper air, causing a glow from time to time. These glow are the aurorae which are also known as the northern lights and southern lights.

1.1.4 Solar corona

Two strong emission lines at λ 5303 Å and 6374 Å were found in the solar atmosphere. Later on some more lines were found in the spectra of the sun. These lines could not be assigned to any of the known atoms or their singly ionized ions. Since the estimated temperature of the photosphere around the sun is about 6000 K, scientists could expect either atoms or their singly ionized ions in the solar atmosphere. In absence of any atom to which these lines could be assigned, scientist coined a name coronium to some unknown atom. No one knew about the physical properties of this coronium, except to say that it was responsible for generation of those unassigned lines. Later on, through laboratory studies at very high temperatures it was found that these unknown lines could be generated by highly ionized ions. For example, the lines at λ 5303 Å and 6374 Å are generated by Fe XIV (thirteen times ionized iron, Fe^{+13}) and Fe X (nine times ionized iron, Fe^{+9}), respectively. Since these highly ionized ions are produced at a temperature of million Kelvin, their was no alternative but to accept that the temperature in the corona around the sun is of million Kelvin.

In spite of energy losses due to radiation as well as conduction, the temperature of corona is found maintained at million Kelvin. It is indeed

a challenging task before the scientists to find out the source for the solar coronal heating. Such coronae are found around a number of stars.

1.1.5 Core of the sun

As an explanation of the source of energy in the sun, it is now well established that in the core, the temperature is of the order of 15 million K and the process of fusion of four hydrogen nuclei into a helium nucleus is going on. Thus, the core of the sun is so hot that the matter there is in the plasma state. Besides the fusion of hydrogen nuclei in a helium nucleus, in some stars, CNO cycle or triple α reactions are going on. Consequently, in the cores of the shining stars, the temperature is sufficiently high to maintain the nuclear reactions. Hence, the material in the cores of stars is in the plasma state.

1.1.6 HII regions

In some parts of the interstellar medium, temperature is so high that the hydrogen gas is in the ionized state. These regions are generally known as the HII regions. These regions are either associated with the evolved stars (stars in the late stage) or they form a big cloud in which the process of star formation is going on.

1.2. Concept of temperature

Let us first understand the meaning of 'temperature' of a gas whose constituent particles may be neutral or may have charge. In a gas in thermal equilibrium, the constituent particles interact (collide) constantly with one another and move with various velocities. The most probable distribution of velocities of these particles is known as the Maxwell-Boltzmann (MB) distribution. Though this velocity distribution is three-dimensional distribution, we however first consider one-dimensional MB distribution expressed as

$$f(u) = A \exp\left(-\frac{1}{2}mu^2/KT\right) \tag{1.1}$$

where u is the velocity deviation of a particle relative to the average velocity, $f(u)\,du$ the number of particles per unit length having velocities

in the range from u to $u+\mathrm{d}u$, m the mass of a particle in the gas, K the Boltzmann constant[2] and T is a physical parameter of the gas, termed as the 'temperature'. The density n, the number of particles per unit length is

$$n = \int_{-\infty}^{\infty} f(u)\ \mathrm{d}u$$

$$= \int_{-\infty}^{\infty} A \exp\Big(-\frac{1}{2}mu^2/KT\Big)\ \mathrm{d}u$$

On defining

$$v_{th} = \sqrt{2KT/m} \qquad \text{and} \qquad y = u/v_{th}$$

we have

$$n = A\int_{-\infty}^{\infty} e^{-y^2} v_{th}\mathrm{d}y$$

$$= Av_{th}\int_{-\infty}^{\infty} e^{-y^2}\mathrm{d}y = Av_{th}\sqrt{\pi} = A\sqrt{\frac{2\pi KT}{m}}$$

Here, A is assumed to be independent of velocity of particles. Thus,

$$A = n\sqrt{\frac{m}{2\pi KT}}$$

A is found to depend on temperature of the gas, and thus, the distribution function (equation 1.1) is

$$f(u) = n\sqrt{\frac{m}{2\pi KT}}\ \exp\Big(-\frac{1}{2}mu^2/KT\Big)$$

Variation of the distribution function $f(u)$ with the velocity u is shown in Figure 1.1 for three temperatures, 100, 120 and 140 K. The figure shows that with the increase of temperature, the number of particles having average velocity (corresponding to $u=0$) decreases, but the width of the graph increases such that total area under the curve is constant, giving the density of particles (*i.e.*, the number of particles per unit length).

[2]We have used K for the Boltzmann constant because k will be used for the wave vector.

In order to understand the meaning of T, we compute mean kinetic energy of the particles averaged over the MB-distribution (1.1) as

$$E_{av} = \frac{\int_{-\infty}^{\infty} \frac{1}{2} m u^2 f(u)\, du}{\int_{-\infty}^{\infty} f(u)\, du}$$

$$= \frac{\int_{-\infty}^{\infty} \frac{1}{2} m u^2 A \exp(-\frac{1}{2} m u^2/KT)\, du}{\int_{-\infty}^{\infty} A \exp(-\frac{1}{2} m u^2/KT)\, du} \tag{1.2}$$

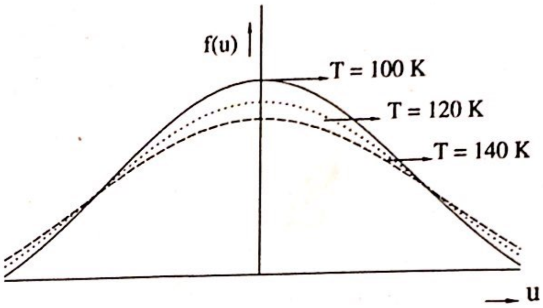


Figure 1.1: Variation of $f(u)$ (in arbitrary units) versus u for three temperatures, 100, 120 and 140 K. With the increase of temperature, the number of particles having average velocity ($u = 0$) decreases, but the width of the profile increases such that total area under the curve is constant, giving the density of particles (*i.e.*, the number of particles per unit length).

Here, we have assumed that the interaction between the constituent particles is very weak, so that the potential energy is neglected. However, the collisions between the particles take place momentarily. Using the relations

$$v_{th} = \sqrt{2KT/m} \qquad \text{and} \qquad y = u/v_{th}$$

equation (1.2) gives

$$E_{av} = \frac{\frac{1}{2} m v_{th}^2 \int_{-\infty}^{\infty} y^2 e^{-y^2} dy}{\int_{-\infty}^{\infty} e^{-y^2} dy}$$

$$= \frac{\frac{1}{2} m v_{th}^2\, (\sqrt{\pi}/2)}{\sqrt{\pi}} = \frac{1}{4} m v_{th}^2$$

$$= \frac{1}{4} m\, \frac{2KT}{m} = \frac{1}{2} KT$$

Thus, for one-dimensional MB distribution, the average kinetic energy is $KT/2$. Thus, the temperature can be defined as the average kinetic per

degree of freedom divided by $(K/2)$. Let us now take three-dimensional MB distribution

$$f(u,v,w) = B\ \exp\Big[-\frac{1}{2}m(u^2+v^2+w^2)/KT\Big] \tag{1.3}$$

where u, v and w are three mutually perpendicular components of velocity. The density n of the gas is

$$n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v,w)\ du\ dv\ dw$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B\ \exp\Big[-\frac{1}{2}m(u^2+v^2+w^2)/KT\Big]\ du\ dv\ dw$$

$$= B\int_{-\infty}^{\infty}\exp(-\frac{1}{2}mu^2/KT)\ du\int_{-\infty}^{\infty}\exp(-\frac{1}{2}mv^2/KT)\ dv$$

$$\times\int_{-\infty}^{\infty}\exp(-\frac{1}{2}mw^2/KT)\ dw$$

$$= B\sqrt{\frac{2\pi KT}{m}}\sqrt{\frac{2\pi KT}{m}}\sqrt{\frac{2\pi KT}{m}} = B\Big(\frac{2\pi KT}{m}\Big)^{3/2}$$

Here, B is assumed to be independent of velocity of particles. Thus,

$$B = n\Big(\frac{m}{2\pi KT}\Big)^{3/2}$$

B is found to depend on temperature of the gas, and the distribution function (equation 1.3) is

$$f(u,v,w) = n\Big(\frac{m}{2\pi KT}\Big)^{3/2}\exp\Big[-\frac{1}{2}m(u^2+v^2+w^2)/KT\Big] \tag{1.4}$$

We compute mean kinetic energy of the particles averaged over this distribution (equation 1.4)

$$E_{av} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}m(u^2+v^2+w^2)f(u,v,w)\ du\ dv\ dw}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u,v,w)\ du\ dv\ dw}$$

$$= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}mu^2\ \exp[-\frac{1}{2}m(u^2+v^2+w^2)/KT]\ du\ dv\ dw}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ \exp[-\frac{1}{2}m(u^2+v^2+w^2)/KT]\ du\ dv\ dw}$$

$$+\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}mv^2\ \exp[-\frac{1}{2}m(u^2+v^2+w^2)/KT]\ du\ dv\ dw}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ \exp[-\frac{1}{2}m(u^2+v^2+w^2)/KT]\ du\ dv\ dw}$$

$$+\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}mw^2\ \exp[-\frac{1}{2}m(u^2+v^2+w^2)/KT]\ du\ dv\ dw}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp[-\frac{1}{2}m(u^2+v^2+w^2)/KT]du\ dv\ dw} \tag{1.5}$$

Let us compute one of the three parts on right side of equation (1.5)

$$y = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2}mu^2 \exp[-\frac{1}{2}m(u^2+v^2+w^2)/KT]\, du\, dv\, dw}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-\frac{1}{2}m(u^2+v^2+w^2)/KT]\, du\, dv\, dw}$$

$$= \frac{\int_{-\infty}^{\infty} \frac{1}{2}mu^2 \exp(-\frac{1}{2}\frac{mu^2}{KT})\, du}{\int_{-\infty}^{\infty} \exp(-\frac{1}{2}\frac{mu^2}{KT})\, du} = \frac{\int_{-\infty}^{\infty} \frac{1}{2}mv_{th}^2 y^2 \exp(-y^2)\, v_{th}\, dy}{\int_{-\infty}^{\infty} \exp(-y^2) v_{th}\, dy}$$

$$= \frac{\frac{1}{2}mv_{th}^2\sqrt{\pi}/2}{\sqrt{\pi}} = \frac{1}{4}mv_{th}^2 = \frac{1}{4}m\sqrt{\frac{2KT}{m}} = \frac{1}{2}\, K\, T$$

Similarly, other two parts would also have the value $KT/2$. Thus,

$$E_{av} = \frac{1}{2}\, KT + \frac{1}{2}\, KT + \frac{1}{2}\, KT = \frac{3}{2}\, KT$$

Hence, a general result is that the average kinetic energy per degree of freedom is $(KT/2)$. As earlier, the temperature can be defined as the average kinetic energy per degree of freedom divided by $(K/2)$.

Temperature and energy in a plasma are so closely related that it is customary to express temperature in the units of energy. One eV energy corresponds to a temperature of 11,605 K.

1.2.1 Simultaneous existence of several temperatures

It is interesting to notice that a plasma may have various kinetic temperatures at the same time. It may happen as the ions and electrons may have independent MB distributions corresponding to different temperatures T_i and T_e of ions and electrons, respectively. This can be understood as the rates of collisions among the electrons and among the ions may be larger than those among the ions and electrons. Thus, each of the species (ions as well as electrons) may have their independent thermal equilibriums. But, this situation of two different temperatures may not last long and they equalize fast.

When the velocities of ions are much smaller than those of the electron, the temperature in the plasma is controlled by the electrons, and hence the MB distribution is given by the electron temperature T_e.

Further, when there is magnetic field, even a single species can have two temperatures. It is because velocity of a particle has two components, one along the magnetic field and the other perpendicular to the

magnetic field. The parallel and perpendicular components of velocities may have their different MB distributions corresponding to the temperatures, $T_{\|}$ and $T_{\perp}$, respectively.

1.2.2 Electron and ion temperatures

There are a number of ways by which plasma can be produced in a laboratory. The electric discharge is the most common method for production of low temperature plasma. Other methods, such as thermal ionization etc. can produce only high temperature plasma, because the temperature is the main cause for the production. A plasma consisting of electrons, ions and neutral atoms shows a non-equilibrium property in the sense that different components are not equally heated. In contrast to an ordinary gas where all the particles have the same mean kinetic energy of thermal motion, in the plasma various components (electrons, ions and neutral atoms) have different mean kinetic energies. As a rule, electrons have much higher energy than the ions and the kinetic energy of ions may be greater than that of the neutral particles.

Since the mean kinetic energies of the three components is different, the plasma would be regarded as having three different temperatures, T_e, T_i, T_a for the three species, such that $T_e > T_i > T_a$. Very large difference between T_e and T_i is a due to the difference between the mass of electron and that of ions and the velocities with which they are moving. The external source of electrical energy which is used to produce and maintain the gas discharge communicates energy direct to the electrons. It is known that the fraction of energy transferred to a body of mass M by another body of mass m when collision takes place is less than $4m/M$ (see discussion in chapter II) and in case of ion-electron collision, it is of the order of $2 \times 10^{-3}/A$, where A is the atomic weight of the element from which the ion is produced.

The electrons must undergo a large number of collisions with the ions before it will loose most of its energy. As the processes in which energy exchange takes place between the electrons and ions occur in simultaneous with the process in which electrons receive energy from the external sources, there is usually a large difference between the electron and ion temperatures. Under some special circumstances, the ion tem-

perature in a highly ionized plasma may be larger than that of electron. Such conditions are found in high powered short lived electric discharges which are used in the studies of controlled thermonuclear discharges. In the arc discharge, the rate of collisions between electrons and ions is quite high and that reduces the temperature difference.

An expression for electron temperature can be derived in the following manner. Suppose E is the axial electric field in the discharge, then the energy gained by an electron per second is eEv_d, where v_d is the drift velocity of the electron. This must be equal to the fraction of energy lost by the electron due to collision. Thus, we have

$$eEv_d = k\left(\frac{1}{2}m_e v_r^2\right)\omega_c \tag{1.6}$$

Here, ω_c is the collision frequency and v_r the random velocity. Now, we have

$$\frac{1}{2}m_e v_r^2 = \frac{3}{2}KT_e \qquad \text{and} \qquad \omega_c = \frac{v_r}{\lambda_e}$$

Here, λ_e is the mean free path of electron in the gas. The drift velocity $v_d = \mu E$, where the mobility $\mu = e/m_e\omega_c$. For these relations, from equation (1.6) we get

$$e\mu E^2 = k\left(\frac{3}{2}KT_e\right)\omega_c \qquad \frac{e^2E^2}{m_e\omega_c} = k\left(\frac{3}{2}KT_e\right)\omega_c$$

$$\frac{e^2E^2}{m_e} = k\left(\frac{3}{2}KT_e\right)\omega_c^2 \qquad \frac{e^2E^2}{m_e} = k\left(\frac{3}{2}KT_e\right)\frac{v_r^2}{\lambda_e^2}$$

$$e^2E^2\lambda_e^2 = k\left(\frac{3}{2}KT_e\right)m_e v_r^2 = k\left(\frac{9}{2}K^2T_e^2\right)$$

Thus, the electron temperature is

$$T_e = \frac{\sqrt{2}\,eE\lambda_e}{3\sqrt{k}K}$$

The mean free path λ_e is inversely proportional to the pressure P of the gas and k can be regarded as $4m/M$ (which is valid only for elastic collisions), then we have

$$T_e = C\,\frac{E}{P}$$

where C is a constant.

Other alternative expressions for electron temperature have been derived by other scientists. However, T_e is found proportional to (E/P) up to 1.5 V/cm mm Hg in argon and neon.

1.3. Debye shielding

A fundamental characteristic of plasma is that it can shield out electric potentials that are applied to it. In order to understand it, let us put an electric field inside a plasma by inserting two charged balls connected to a battery (Figure 1.2). Here, we assume that a layer of dielectric restricts the plasma from recombining on the surface or the battery is large enough to maintain the potential in spite of recombination. Now, the balls almost immediately would attract particles of opposite charge and a cloud of positive ions would be formed around the negative ball and a cloud of electrons would be formed around the positive ball.

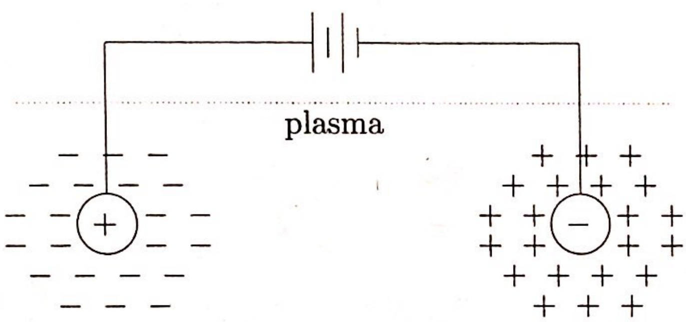


Figure 1.2: Shows two charged balls connected to a battery and inserted inside a plasma

In case the plasma is cold and there are no thermal motions of charged particles, the surrounding cloud in each case would have as many charges as there are in the ball. Thus, in the plasma, outside the clouds, there would be no electric field and the shielding is prefect. On the other hand, if the plasma temperature is finite, the particles at the edge of the cloud (there electric field is weak), have enough thermal energy to escape from the electrostatic potential well. The edge of the cloud is estimated to occur at the radial distance where the potential energy is approximately equal to the thermal kinetic energy KT of the particles. Consequently, the shielding is not complete.

Let us calculate the approximate thickness of such a charge cloud. Suppose the potential ϕ on the plane $x = 0$ is kept at a value ϕ_0 by a perfectly transparently grid (Figure 1.3). Here, x is the distance measured radially. Now, our object is to compute $\phi(x)$. For simplicity, we assume that the ratio M/m_e (ratio of the mass of positively charged ion and that of election) is very high (infinite in the mathematical language), so that the positively charged ions do not move but form a uniform background of positive charge through which the negatively charged electrons are moving in the gas. Since the situation is spherically symmetric, we can account for the radial variation. Thus, we consider one-dimensional Poisson equation

$$\epsilon_0 \nabla^2 \phi = -q \qquad \epsilon_0 \frac{d^2\phi}{dx^2} = -e(n_i - n_e) \tag{1.7}$$

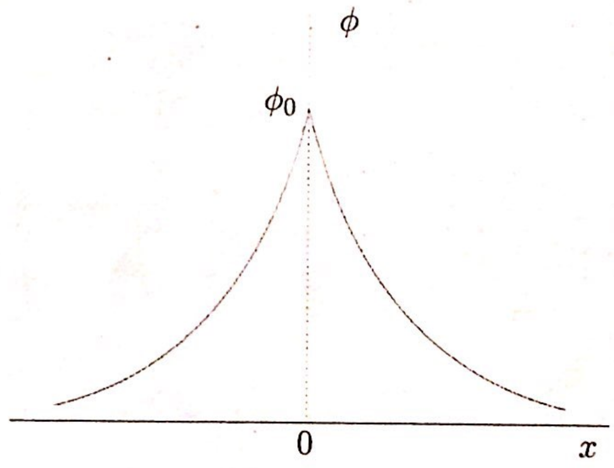


Figure 1.3: Variation of potential as a function of distance.

Here, we have accounted for the hydrogen plasma, and hence the positively charged ions are protons. At large distance, density of protons is equal to that of the electrons and is denoted by n. As the protons are not moving, the density of protons n_i is n everywhere. Thus,

$$n_i = n \tag{1.8}$$

In the presence of a potential energy $q\phi(\equiv -e\phi)$, the electron distribution function is

$$f(u) = A \exp\Big[-\Big(\frac{1}{2}m_e u^2 - e\phi\Big)/KT_e\Big] \tag{1.9}$$

where T_e is the electron temperature, u the velocity of electron. Electron density is

$$\begin{aligned} n_e &= \int_{-\infty}^{\infty} f(u)\, \mathrm{d}u \\ &= \int_{-\infty}^{\infty} A \exp\Big[-\Big(\frac{1}{2}m_e u^2 - e\phi\Big)/KT_e\Big]\, \mathrm{d}u \\ &= A \exp(e\phi/KT_e) \int_{-\infty}^{\infty} \exp\Big(-\frac{1}{2}m_e u^2/KT_e\Big)\, \mathrm{d}u \\ &= A \exp(e\phi/KT_e) \sqrt{\frac{2\pi KT_e}{m_e}} \end{aligned} \tag{1.10}$$

At large distance

$$x \to \infty \qquad \phi \to 0 \qquad n_e \to n$$

$$n = A \sqrt{\frac{2\pi KT_e}{m_e}}$$

Thus, equation (1.10) becomes

$$n_e = n \exp(e\phi/KT_e) \tag{1.11}$$

Using equations (1.8), (1.9), and (1.11) in (1.7), we get

$$\begin{aligned} \epsilon_0 \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} &= -e\Big[n - n\exp(e\phi/KT_e)\Big] \\ &= en\Big[\exp(e\phi/KT_e) - 1\Big] \\ &= en\Big[1 + \frac{e\phi}{KT_e} + \frac{1}{2!}\Big(\frac{e\phi}{KT_e}\Big)^2 + \ldots - 1\Big] \\ &= en\Big[\frac{e\phi}{KT_e} + \frac{1}{2!}\Big(\frac{e\phi}{KT_e}\Big)^2 + \ldots\Big] \end{aligned}$$

Thus,

$$\epsilon_0 \frac{d^2\phi}{dx^2} = \frac{ne^2}{KT_e}\phi \qquad\qquad \frac{d^2\phi}{dx^2} = \frac{\phi}{\lambda_D^2} \tag{1.12}$$

where

$$\lambda_D = \left(\frac{\epsilon_0 KT_e}{ne^2}\right)^{1/2}$$

Solution of equation (1.12) is

$$\phi = C\ \exp\Big(-\,|x|/\lambda_D\Big) + D\ \exp\Big(|x|/\lambda_D\Big)$$

Second part on the right side of this equation is not feasible as it shows unphysical situation of increase of potential with the increase of distance. Hence, the constant D is substituted equal to zero. Thus, we have

$$\phi = C\ \exp\Big(-\,|x|/\lambda_D\Big) \tag{1.13}$$

At $x = 0$, we have $\phi = \phi_0$. Thus, $C = \phi_0$, and equation (1.13) becomes

$$\phi = \phi_0\ \exp\Big(-\,|x|/\lambda_D\Big)$$

The parameter λ_D is known as the *Debye length* and is a measure of the shielding distance or of the thickness of the sheath. Thus, Debye length is the distance at which the potential drops by a factor e.

1.4. Fundamental concepts

In this section, we shall discuss about some basic properties of plasma.

1.4.1 Kinetic pressure in a partially ionized gas

For an unionized gas having n particles per unit volume at temperature T, the pressure is $P = nKT$. On partial ionization, the gas generally has three components: (i) electrons, (ii) positive ions and (iii) neutral particles. Suppose x denotes the fraction of particles ionized. For hydrogen plasma, in a unit volume, there would be nx electrons and nx protons and $n(1-x)$ neutral particles. At a kinetic temperature T, the pressure exerted by these species is $nxKT$, $nxKT$, and $n(1-x)KT$,

respectively. Following the Dalton's law for partial pressures, the total pressure P_t of the ionized gas is

$$\begin{aligned} P_t &= P_e + P_i + P_a \\ &= nxKT + nxKT + n(1-x)KT \\ &= n(1+x)KT = (1+x)P \end{aligned}$$

It shows that on ionization, total pressure of gas increases. For fully ionized gas we have $x = 1$ and total pressure becomes two times that of the neutral gas. That is, $P_t = 2nKT$.

1.4.2 Mean free path and collision cross section

Following kinetic theory of gases, the mean free path is defined as the mean distance traveled by a particle between two successive collisions and is denoted by λ. The time τ required to cover the mean free path is known as the *mean free time*. The reciprocal of mean free time, gives the number of collisions the particles occurred per unit time and is known as the collision frequency denoted by ν. Thus, for the random velocity v of particles, we have $\lambda = v/\nu$. The parameters λ, τ and ν can be related to the characteristics which determine the collision process between the particles by introducing the concept of effective collision cross section.

In the classical scenario, a collision is said to occur when the distance between the centers of colliding particles is less than $2a$ where a is the radius of each particle, assumed to be a rigid sphere. The quantity $2a$ is known as the effective collision radius and the quantity $\pi(2a)^2 = 4\pi a^2$ is known as the effective collision cross section. Assuming Maxwell-Boltzmann distribution, the kinetic theory of gases shows that the mean free path is

$$\lambda = \frac{1}{4n\pi a^2} = \frac{1}{n\sigma}$$

where σ denotes the effective collision cross section.

For obtaining the mean free path in a plasma, we have to account for the interaction between charged particles in it. A partially ionized plasma has three components: (i) electrons, (ii) positive ions and (iii) neutral particles. For neutral particles, short range forces are acting

among them only when the particles approach each other within a distance of 10^{-10} to 10^{-9} m. In case of the charged particles, the forces among them have very long ranges and are appreciable even at very large distances. Thus, in a plasma, Coulomb forces play dominant part. In a plasma, each charged particle is in the field of remaining charges particles (electrons and ions). Due to random motions of ions and electrons, this field is subject to continuous variation in magnitude as well as in direction.

This plasma field gives rise to a continuous variation in magnitude and in direction of the velocity of a charged particle in the plasma. Due to electrostatic interaction between the charged particles, it can be found that the gradual change in the direction of velocity is the result of a large number of weak interactions. In considering the charged particles collisions, one has to define what kind of interaction can be regarded as collision. A collision is generally defined as an interaction which leads to a deflection or scattering of a particle in the Coulomb field of other particle through a large angle, say 90^o or more. Further the deflection of 90^o can be achieved through a large number of encounters which a single particle may suffer due to multiple interactions of the single particle with many other particles. This is known as the long range encounter. In a partially ionized plasma, let us consider an electron as a test particle moving through the ionized gas.

The test electron will start its motion in a direction and after some successive deflections due to Coulomb field of ions, say, the electron will suffer a total deflection of 90^o or more. The average time required for such a process to take place is known as the electron-ion collision time. In the same manner, we define an electron-electron collision time and the electron-neutral time. If ν_{ei}, ν_{ee} and ν_{ea} denote the frequency of collisions of an electron per unit time with ions, electrons and neutral particles, respectively, the total frequency ω_c of collision is

$$\omega_c = \nu_{ei} + \nu_{ee} + \nu_{ea}$$

Without considering the distribution of velocities it can be shown from the kinetic theory that the particle cross section σ_c is related to the

mean free path λ through the relation

$$\sigma_c = \frac{1}{n\lambda}$$

where n denotes the particle density. If Q is the cross section of all the particles of the gas then we have

$$Q = n\sigma_c = \frac{1}{\lambda}$$

where the total effective cross section Q will depend upon n the number of particles per unit volume and thus upon the pressure and temperature of the gas. Whenever a collision between an electron and a gas particle takes place, it may be elastic collision or inelastic one, resulting in the excitation or ionization. If P_{el}, P_{ion} and P_{ex} denote the probabilities for elastic, ionization, excitation processes, respectively, then

$$\begin{aligned} Q &= P_{el}q + P_{ion}q + P_{ex}q \\ &= Q_e + Q_i + Q_{ex} \end{aligned}$$

This simple relation showing addition is only possible as long as the state of the gas remains substantially unchanged, *i.e*, the total number of the collision products is small as compared to the number of particles present and as long as the processes are independent of one another.

1.4.3 Mobility of charged particles

For a neutral gas, we are concerned with the random velocity distribution. But in a plasma, besides this random velocity, the constituent particles move as a whole in the direction of the applied electric field, depending on the nature of charge, and thus have a drift velocity. The drift velocity acquired per unit field gradient is known as the mobility.

Let us consider the case where an electron is acted upon by an alternating e.m.f. $E = E_0 \cos \omega t$, where E_0 is the magnitude and ω the angular frequency. When we consider the collision of electrons with gas particles and if f is the restoring force per unit drift velocity then the equation of motion of electron is

$$m\,\frac{dv_d}{dt} + f v_d = eE_0\, e^{i\omega t} \qquad \text{or} \qquad \frac{dv_d}{dt} + \omega_c v_d = \frac{e}{m}\, E_0\, e^{i\omega t}$$

where $\omega_c = f/m$ is the collision frequency of the electron with gas particles. Assuming $v_d = A\, e^{i\omega t}$, where A is a constant, we get

$$A(\omega_c + i\omega) = \frac{e}{m}\, E_0 \qquad \text{or} \qquad A = \frac{eE_0}{m(\omega_c + i\omega)}$$

and the drift velocity is

$$v_d = \frac{eE_0}{m(\omega_c + i\omega)}\, e^{i\omega t}$$

For a d.c. field we have

$$A = \frac{eE_0}{m\omega_c} \qquad \text{and} \qquad v_d = \frac{eE_0}{m\omega_c}$$

and mobility

$$\mu = \frac{v_d}{E_0} = \frac{e}{m\omega_c}$$

1.4.4 Effect of magnetic field on mobility of electrons

Let in an electric field $E = E_0 \cos\omega t$ acting along x-axis, electrons are moving with a velocity $\vec{v} = v_x\hat{i} + v_y\hat{j}$. Suppose, we apply a transverse magnetic field H along z-axis. Then the equation of motion of electron is

$$m\frac{d\vec{v}}{dt} + m\omega_c\vec{v} + e\vec{v} \times \vec{H} = \vec{E}$$

Thus, we have

$$\frac{dv_x}{dt} + \omega_c v_x + \omega_H v_y = \frac{e}{m} E_0\, e^{i\omega t} \tag{1.14}$$

$$\frac{dv_y}{dt} + \omega_c v_y - \omega_H v_x = 0 \tag{1.15}$$

where $\omega_H = eH/m$ is known as the *electron cyclotron frequency*. If we assume that $v_x = A\, e^{i\omega t}$ and $v_y = B\, e^{i\omega t}$ equation (1.15) gives

$$B(i\omega + \omega_c) = A\omega_H \qquad B = \frac{A\omega_H}{(i\omega + \omega_c)}$$

then equation (1.14) gives

$$A\left(\omega_c + i\omega + \frac{\omega_H^2}{(\omega_c + i\omega)}\right) = \frac{e}{m}E_0 \qquad A = \frac{eE_0/m}{\omega_c + i\omega + \omega_H^2/(\omega_c + i\omega)}$$

Thus, we have

$$v_x = \frac{eE_0(\omega_c + i\omega)\ e^{i\omega t}}{m[(\omega_c + i\omega)^2 + \omega_H^2]}$$

For d.c. field, $\omega = 0$ and we have

$$v_x = \frac{eE_0\omega_c}{m[\omega_c^2 + \omega_H^2]}$$

and mobility

$$\mu_H = \frac{v_x}{E_0} = \frac{e\omega_c}{m[\omega_c^2 + \omega_H^2]}$$

$$= \frac{e}{m\omega_c[1 + \omega_H^2\tau^2]} = \frac{\mu}{1 + \omega_H^2\tau^2}$$

where μ is the mobility in absence of the magnetic field and $\tau = 1/\omega_c$ the time between successive collisions between the electron and neutral particles. It shows that in the presence of magnetic field, the mobility of electrons is reduced in a direction perpendicular to the magnetic field.

For ions, almost similar expression can be derived. But since mass of ions is much larger than that of electron, the effect of magnetic field on the mobility of ions is very small.

1.4.5 Thermal conductivity

In a thermonuclear plasma, there exists temperature gradient and the energy is lost through thermal conduction. The temperature is always high at the center of the plasma than at its surface and therefore there is a radial flow of energy. Taking x-axis along the temperature gradient, the energy flux is

$$\phi = -\overline{K}\ \frac{\mathrm{d}T}{\mathrm{d}x}$$

where $\overline{K}$ denotes the thermal conductivity of plasma. The number of particles flowing through the origin due to collisions between x and $x+\mathrm{d}x$ is

$$\mathrm{d}n_c = \frac{v(x)}{6\lambda}\ n\ \mathrm{e}^{-x/\lambda}\ \mathrm{d}x$$

Here, we assume uniform density but the velocity v of the particles is a function of x due to the fact that temperature which is a function of

velocity has a spatial variation. As in the case of diffusion, the energy of each particle is $m[v(x)]^2/2$ and energy flux in the region between x and $x + \mathrm{d}x$ is

$$\mathrm{d}\phi = \frac{m}{12\lambda}[v(x)]^3 n\, \mathrm{e}^{-x/\lambda}\, \mathrm{d}x$$

The net energy flow in the x direction is

$$\phi = \frac{mn}{12\lambda}\Big[\int_{-\infty}^{0}[v(x)]^3 \mathrm{e}^{x/\lambda}\, \mathrm{d}x - \int_{0}^{\infty}[v(x)]^3 \mathrm{e}^{-x/\lambda}\, \mathrm{d}x\Big]$$

Here, the first integral represents the incoming flux whereas the second integral represents the outgoing flux. Expressing

$$v(x) = v_0 + \frac{\mathrm{d}v}{\mathrm{d}x}x$$

we get

$$[v(x)]^3 = v_0^3 + 3xv_0^2\,\frac{\mathrm{d}v}{\mathrm{d}x}$$

and therefore

$$\phi = \frac{mn}{12\lambda}\Big[v_0^3\int_{-\infty}^{0}\mathrm{e}^{x/\lambda}\, \mathrm{d}x - v_0^3\int_{0}^{\infty}\mathrm{e}^{-x/\lambda}\, \mathrm{d}x + \int_{-\infty}^{0}3xv_0^2\,\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{e}^{x/\lambda}\, \mathrm{d}x$$
$$- \int_{0}^{\infty}3xv_0^2\,\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{e}^{-x/\lambda}\, \mathrm{d}x\Big]$$

Since

$$\int_{-\infty}^{0}\mathrm{e}^{x/\lambda}\, \mathrm{d}x = \lambda \qquad \text{and} \qquad \int_{0}^{\infty}\mathrm{e}^{-x/\lambda}\, \mathrm{d}x = \lambda$$

we have

$$\phi = \frac{mnv_0^2}{4\lambda}\,\frac{\mathrm{d}v}{\mathrm{d}x}\Big[\int_{-\infty}^{0}x\, \mathrm{e}^{x/\lambda}\, \mathrm{d}x - \int_{0}^{\infty}x\, \mathrm{e}^{-x/\lambda}\, \mathrm{d}x\Big]$$

$$= \frac{mnv_0^2}{4\lambda}\,\frac{\mathrm{d}v}{\mathrm{d}x}2\lambda^2 = \frac{mnv_0^2\lambda}{2}\,\frac{\mathrm{d}v}{\mathrm{d}x}$$

When the velocity distribution of particles is Maxwellian, we have

$$\frac{1}{2}\,mv^2 = \frac{1}{2}\,KT$$

so that

$$2mv_0\Big(\frac{\mathrm{d}v}{\mathrm{d}x}\Big)_0 = K\frac{\mathrm{d}T}{\mathrm{d}x}$$

and therefore

$$\phi = -\frac{nv\lambda K}{4}\frac{\mathrm{d}T}{\mathrm{d}x} = -\overline{K}\ \frac{\mathrm{d}T}{\mathrm{d}x}$$

where

$$\overline{K} = \frac{nv\lambda K}{4}$$

As discussed earlier, using the relation between scattering cross section σ_c and λ as $\sigma_c = 1/n\lambda$, we have

$$\overline{K} = \frac{vK}{4\sigma_c}$$

1.4.6 Dielectric constant of plasma

Suppose plasma is subject to an oscillating electric field $\vec{E}=\vec{E_0}\ \mathrm{e}^{i\omega t}$, where ω is frequency of the applied field. Due to this field a current flows in the plasma and when collisions of charged particles with neutral atoms are neglected, the equation of motion is

$$m\ \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = e\ \vec{E_0}\ \mathrm{e}^{i\omega t}$$

For non-relativistic motion, mass of the particle remains constant and integration of this equation gives

$$\vec{v} = \frac{e\vec{E_0}}{mi\omega}\ \mathrm{e}^{i\omega t} = \frac{e\vec{E}}{mi\omega} \tag{1.16}$$

For convenience, the constant of integration is assumed to be zero. If n is the number of particles per unit volume, the current density is

$$\vec{i_c} = ne\vec{v} = \frac{ne^2\vec{E}}{mi\omega} \tag{1.17}$$

From Maxwell equation, we have

$$\nabla\times\ \vec{B} = \mu_0\Big(i_c + \epsilon_0\frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\Big)$$

Using expression for $\vec{E}$ and equation (1.17) here, we get

$$\nabla\times\ \vec{B} = \mu_0\Big(\frac{ne^2\vec{E}}{mi\omega} + \epsilon_0 i\omega\vec{E}\ \Big) = \mu_0\epsilon_0 j\omega\vec{E}\ \Big(1 - \frac{\omega_p^2}{\omega^2}\Big) \tag{1.18}$$

where $\omega_p = \sqrt{ne^2/m\epsilon_0}$ is the electron plasma frequency. If ϵ is the dielectric constant, we have

$$\nabla\times \vec{B} = \epsilon_0\epsilon\mu_0 \frac{d\vec{E}}{dt} = \epsilon_0\epsilon\mu_0 i\omega \vec{E} \tag{1.19}$$

From equations (1.18) and (1.19), the dielectric constant ϵ is

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2}$$

This equation shows that when $\omega_p < \omega$, the dielectric constant is positive and the wave can propagate. But when $\omega_p > \omega$, the dielectric constant is negative and the wave cannot propagate. The cut off takes place when $\omega_p = \omega$.

Inclusion of collisions

When collisions of electrons with neutral atoms are taken into account, the equation of motion is

$$m\,\frac{d\vec{v}}{dt} + m\omega_c\vec{v} = e\vec{E}_0\, e^{i\omega t}$$

where ω_c is the collision frequency for momentum transfer. When we take $\vec{v}=\vec{A}\, e^{i\omega t}$, we have

$$mi\omega\vec{A}\, e^{i\omega t} + m\omega_c\vec{A}\, e^{i\omega t} = e\vec{E}_0\, e^{i\omega t}$$

so that

$$\vec{A} = \frac{e\vec{E}_0}{m(i\omega + \omega_c)}$$

and

$$\vec{v} = \frac{e\vec{E}_0\, e^{i\omega t}}{m(i\omega + \omega_c)} = \frac{e\vec{E}\,(\omega_c - i\omega)}{m(\omega_c^2 + \omega^2)}$$

Therefore, we have

$$\vec{i}_c = ne\vec{v} = \frac{ne^2\vec{E}\,(\omega_c - i\omega)}{m(\omega_c^2 + \omega^2)}$$

and

$$\nabla\times\vec{B}=\mu_0\Big(i_c+\epsilon_0\frac{\mathrm{d}\vec{E}}{\mathrm{d}t}\Big)$$

$$=\mu_0\Big(\frac{ne^2\vec{E}\,(\omega_c-i\omega)}{m(\omega_c^2+\omega^2)}+\epsilon_0 i\omega\vec{E}\,\Big)$$

$$=\mu_0\epsilon_0 i\omega\vec{E}\,\Big(1+\frac{\omega_p^2(\omega_c-i\omega)}{i\omega(\omega_c^2+\omega^2)}\Big) \tag{1.20}$$

The dielectric constant is now obviously complex and therefore

$$\nabla\times\vec{B}=\epsilon_0\mu_0(\epsilon'-i\epsilon'')\frac{\mathrm{d}\vec{E}}{\mathrm{d}t}=\epsilon_0\mu_0(\epsilon'-i\epsilon'')i\omega\vec{E} \tag{1.21}$$

where ϵ' and ϵ'' are real and imaginary parts of the dielectric constant. From equations (1.20) and (1.21), we have

$$\epsilon'=1-\frac{\omega_p^2}{\omega_c^2+\omega^2} \qquad \epsilon''=\frac{\omega_p^2\omega_c}{\omega(\omega_c^2+\omega^2)}$$

Now, we have

$$\frac{\mathrm{d}\epsilon''}{\mathrm{d}\omega_c}=\frac{(\omega_c^2+\omega^2)-2\omega_c^2}{(\omega_c^2+\omega^2)}=0$$

It gives $\omega_c=\omega$. It is therefore clear that when n the number of electrons per unit volume can be considered as constant with regard to variation of pressure, the loss ϵ'' becomes maximum when $\omega=\omega_c$.

1.4.7 Optical properties of plasma

Relation between dielectric and optical properties of gases was derived long back. Later on such relation for plasma has been obtained. In the preceding section we have obtained dielectric constant ϵ for plasma as

$$\epsilon=1-\frac{\omega_p^2}{\omega^2}$$

where $\omega_p=\sqrt{ne^2/m\epsilon_0}$ is the plasma frequency. Thus, the refractive index μ of the plasma is

$$\mu=\sqrt{1-\frac{\omega_p^2}{\omega^2}}=\sqrt{1-\frac{n}{n_c}}$$

where n_c is the electron density which gives the upper limit below which the radiation of frequency ω cannot penetrate the plasma. When ω_p is smaller than ω, the radiation flows through the plasma. As the refractive index is less than one, the velocity of light in the plasma exceeds the speed of light in vacuum. When ω_p is larger than ω, the refractive index becomes imaginary and therefore the incident radiation are reflected back from the plasma. When $\omega_p = \omega$, the refractive index is zero, which is true only in case of an ideal plasma where collisions are not accounted for. When collisions are accounted for, the dielectric constant $\epsilon = (\epsilon' - i\epsilon'')$ is

$$\epsilon = 1 - \frac{\omega_p^2}{\omega_c^2 + \omega^2} - i\frac{\omega_p^2\omega_c}{\omega(\omega_c^2 + \omega^2)}$$

Thus, the refractive index μ of the plasma is

$$\mu^2 = 1 - \frac{\omega_p^2}{\omega_c^2 + \omega^2}\left[1 + i\frac{\omega_c}{\omega}\right]$$

1.5. Problems and questions

1. Show that in Maxwellian distribution, average kinetic energy per degree of freedom is $(KT/2)$. Discuss the concept of temperature.
2. Write a short note on natural plasma.
3. What is Debye shielding? Derive an expression for Debye length.
4. Write short notes on the following

   (i) Plasma in ionosphere

   (ii) Plasma in van Allen belts

   (iii) Plasma in aurorae

   (iv) Plasma in solar corona

   (v) Plasma in the core of the sun

   (vi) Debye length

   (vii) Debye shielding

   (viii) Mobility of charged particles

(ix) Kinetic pressure in a partially ionized plasma

(x) Mean free path and collisional cross section in a plasma

2

Production of Plasma

For production of plasma, atoms in a gas are to be ionized, so that we get positively charged ions and electrons. For single ionization, the number of positively charged ions is equal to that of electrons. So, stating from a gas having one kind of atoms, we can have three species: (i) atoms, (ii) positively charged ions and (iii) electrons. For ionization of an atom, the minimum amount of energy required is known as the ionization potential of the atom. The energy may be supplied to an atom through various processes and some of them are as the following:

(i) Through collisions with external electrons

(ii) Through absorption of a photons of proper frequency

(iii) By heating the gas

(iv) Through high intensity laser

In this chapter, we shall discuss about the production of plasma by using the above processes.

2.1 Production of plasma through collisions

In an atom, generally, all electrons are not in the ground state. Some of them are in the excited state and the majority of them is in the ground state. When some amount of energy is supplied to the atom through some process, an electron in the atom may go to one of the higher states (this process is known as the excitation) or may go out of the atom (this process is known as the ionization). In the excitation process, an electron in the ground or in an excited state after absorbing energy moves to an outer state. In case this energy is supplied by an external photon, the

energy difference between the initial and final states of the electron must be equal to the energy of the photon as well as the transition must be radiatively allowed. Depending on the energy supplied, the atom can be excited to any of the outer states. The amount of energy suppled in the excitation process is known as the excitation potential. In general, the life-time of an excited is about $10^{-7} - 10^{-8}$ s. (Life time of a metastable level is 10^{-3} s.) Thus, an atom returns back quickly to a lower state. It may lead to a cascading of electron and finally the electron comes to the ground state. The process of going to lower energy states is known as the excitation. In each deexcitation, a photon of energy equivalent to the energy difference between the two states is emitted.

When the energy supplied to the atom is sufficient to detach an electron from the atom, the process is known as the ionization of the atom and the detached electron becomes free. The minimum energy required to detach an electron from an atom is known as the ionization potential of the atom. After one ionization, if the ionized atom has other electron, second electron can also detached and the minimum amount of energy supplied is known as the second ionization potential. The energy of successive ionization goes on increasing.

One of the processes of supplying energy may be collision by electron. When an external electron collides with an atom, it may give up a fraction of its kinetic energy to one of the electrons in the atom. After giving energy the external electron moves with a decreased energy and in a deviated path. During the collision, the energy is supplied to one of the electrons in the atom. Because of exchange of energy between target (atom) and projectile (electron), the process is known as inelastic collision. (In an elastic collision, there is no exchange of energy between target and projectile.) Remember that for the collisional transitions, there are not selection rules. All the transitions are possible.

Suppose an electron of mass m moving with velocity $\vec{v}$ collides with an atom of mass M at rest. After collision, the electron and atom travel at angles θ and ϕ with respect to the initial direction of the electron with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively (Figure 2.1). Since the particles are moving without any interaction between them, there is no potential

energy. The conservation of kinetic energy gives

$$\frac{1}{2}\,mv^2 = \frac{1}{2}\,mv_1^2 + \frac{1}{2}\,Mv_2^2 \tag{2.1}$$

Conservation of linear momentum along and perpendicular to the initial direction of motion of electron is, respectively,

$$mv = mv_1 \cos\theta + Mv_2 \cos\phi \tag{2.2}$$

and

$$0 = mv_1 \sin\theta - Mv_2 \sin\phi \tag{2.3}$$

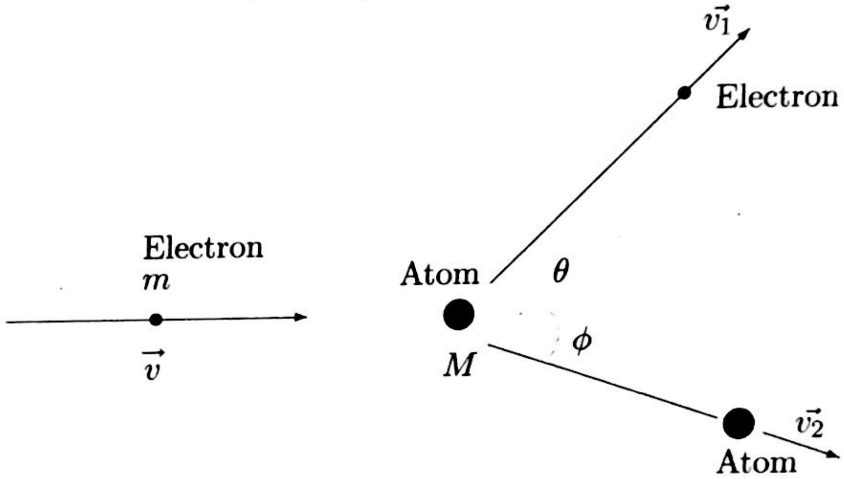


Figure 2.1: Electron of mass m moving with a velocity $\vec{v}$ collides with an atom of mass M at rest. After collision, the electron moves with a velocity $\vec{v_1}$ in a direction θ and the atom moves with a velocity $\vec{v_2}$ in a direction ϕ with respect to the direction of the incident electron.

Equations (2.2) and (2.3), can be rearranged as

$$mv - Mv_2 \cos\phi = mv_1 \cos\theta$$

and

$$Mv_2 \sin\phi = mv_1 \sin\theta$$

On squaring these equation and adding, we get

$$m^2v^2 + M^2v_2^2 - 2mMvv_2 \cos\phi = m_1^2v_1^2 \tag{2.4}$$

Equation (2.1) can be expressed as

$$m^2v^2 - Mmv_2^2 = m^2v_1^2 \tag{2.5}$$

Subtracting equation (2.5) from (2.4), we have

$$M(M+m)v_2^2 = 2Mmvv_2\cos\phi \qquad v_2 = \frac{2m\cos\phi}{(M+m)}v$$

The energy given to the atom E_a is

$$E_a = \frac{1}{2}Mv_2^2 = \frac{1}{2}M\frac{4m^2v^2\cos^2\phi}{(M+m)^2} = \frac{4mM\cos^2\phi}{(M+m)^2}E_e$$

where $E_e = mv^2/2$ is the energy of incident electron. Mass of electron is much smaller than that of atom and taking $\phi = 0$, we have

$$E_a = \frac{4m}{M}E_e$$

The quantity $4m/M$ is often denoted by $\mathcal{R}$ and is the energy lost by electron per collision. For hydrogen atom, we have $M = 1836\ m$ and thus for each collision with hydrogen atom, an electron can loose 1/459 of its initial energy. For collision with other atoms, the fractional loss of incident energy is much smaller. Hence, the energy loss due to collision is only a small fraction of its initial energy. Since the loss is proportional to the initial energy, if an electron strikes with large energy, a state may arrive when the energy transferred is sufficient for ionization of the atom. At further energies, the energy above the ionization potential would be carried out by the emitted electron. With the emission of electron, plasma is generated.

2.1.1 Townsend theory for collisional ionization

Utilizing the process of collision of electrons with atoms, Townsend developed a theory as the following. Suppose an electric field E is applied between two parallel plates separated by a distance d. One of them is cathode, emitting electrons. At a distance x from the cathode, suppose n be the number of electrons crossing per unit area per unit time. In a thickness dx, the number of electrons is n dx and the number of ions produced by the electrons is proportional to n dx. Suppose dn be the number of ions produced by these electrons, then we have

$$\mathrm{d}n \propto n\ \mathrm{d}x \qquad \mathrm{d}n = \alpha n\ \mathrm{d}x$$

where α is known as the *Townsend's first ionization coefficient.* Here, we have neglected diffusion and recombination of electrons. The value of α depends on pressure P in the medium and the electric field E between the plates. Integration of above equation, the number of electrons on the opposite plate (anode) is

$$n = n_0 \, e^{\alpha d}$$

where n_0 is the number of electrons per unit surface area produced at the cathode. If i_0 is the initial current at the cathode and i the current at the anode, then we have

$$\frac{i}{i_0} = \frac{n}{n_0} = e^{\alpha d}$$

Townsend made an assumption that the energy gained by the electron in moving a distance l must be equal to the ionization potential V_i of the gas. Thus, we have

$$eEl = eV_i \qquad\qquad l = \frac{V_i}{E}$$

The probability that the electron will move a distance l without collision is $\exp(-l/\lambda)$, where λ is the mean free path of the electron in the gas. Thus, the number of collisions per unit length is $\exp(-l/\lambda)/\lambda$. If we assume that each collision will result in ionization then the number of ions produced per cm per electron which is defined as α is

$$\alpha = \frac{e^{-l/\lambda}}{\lambda} = \frac{e^{-(V_i/E\lambda)}}{\lambda}$$

Since $\lambda = L/P$, where L is the mean free path of the electron in the gas at one atm and P the pressure of the gas. Then we have

$$\alpha = \frac{e^{-(V_i P/EL)}}{L/P} \qquad\qquad \frac{\alpha}{P} = \frac{1}{L} \, e^{-(V_i P/EL)}$$

Townsend introduced two constants $A = 1/L$ and $B = V_i/L$, which depend on the gas. Thus, we have

$$\frac{\alpha}{P} = A \exp\left(- \frac{B}{E/P} \right) \tag{2.6}$$

Townsend then argued that had this been the only process, the variation of current with voltage would have been represented by $i = i_0 \, e^{\alpha d}$.

But, actually the current increases very rapidly beyond a certain applied voltage. Townsend made an assumption that at this stage the secondary electrons are produced by the positive ions themselves as they impinge on the cathode surface. Besides the positive ions, photons incidence on the cathode surface can also produce secondary electrons.

For $n = n_0\, e^{\alpha d}$, the number of electrons produced by a single electron is $e^{\alpha d}$ and the number of positive ions produces $(e^{\alpha d} - 1)$.[1] These positive ions move towards the cathode and incident on it. Suppose γ is the number of electrons released from the cathode due to the incidence of a single positive ion, then the number of electrons produced by the positive ions is $\gamma(e^{\alpha d} - 1)$. These electrons will produce $\gamma(e^{\alpha d} - 1)^2$ electrons due to collisional ionization in the gas. At the same time $\gamma(e^{\alpha d} - 1)^2$ positive ions will incident on the cathode surface and will produce $\gamma^2(e^{\alpha d} - 1)^2$ electrons. These electrons will now produce $\gamma^2(e^{\alpha d} - 1)^3$ electrons in the gas. This process will continue and the total number of electrons will be

$$\begin{aligned} & e^{\alpha d} + \gamma(e^{\alpha d} - 1)e^{\alpha d} + \gamma^2(e^{\alpha d} - 1)^2 e^{\alpha d} + \ldots \text{up to infinity} \\ &= e^{\alpha d}[1 + \gamma(e^{\alpha d} - 1) + \gamma^2(e^{\alpha d} - 1)^2 + \ldots \text{up to infinity}] \\ &= \frac{e^{\alpha d}}{1 - \gamma(e^{\alpha d} - 1)} \end{aligned} \tag{2.7}$$

γ introduced by Townsend is a measure of secondary ionization and is known as the *Townsend's second ionization coefficient.*

2.1.2 Breakdown potential

It is found experimentally that at a certain value of applied potential, the current flowing has a tendency to increase to an extremely high value. Equation (2.7) indicates that the current tends to become infinite when the denominator approaches to zero provided that the assumptions made are valid. This condition physically represents the breakdown of the gas under the applied potential. Thus, the criterion for the breakdown is

$$1 - \gamma(e^{\alpha d} - 1) = 0; \qquad \gamma = \frac{1}{(e^{\alpha d} - 1)}; \qquad \ln\left(1 + \frac{1}{\gamma}\right) = \alpha d \tag{2.8}$$

[1] $e^{\alpha d}$ is the total number of electrons which include the electron which has produced them. Hence, the total number of electrons generated is $(e^{\alpha d} - 1)$ and the same number of positive ions are produced.

From equations (2.6) and (2.8), we have

$$\frac{1}{Pd}\ln\left(1+\frac{1}{\gamma}\right) = A\exp\left(-\frac{B}{E/P}\right)$$

$$\ln\left(1+\frac{1}{\gamma}\right) = APd\exp\left(-\frac{B}{E/P}\right)$$

Taking logarithm of this equation, we have

$$\frac{BP}{E} = \ln\left(\frac{A}{\ln(1+1/\gamma)}\right) + \ln(Pd)$$

For total potential difference V_s between two electrodes for the breakdown, we have $E = V_s/d$ and using $C = \ln\left(\frac{A}{\ln(1+1/\gamma)}\right)$, we get

$$V_s = \frac{B(Pd)}{C+\ln(Pd)} \tag{2.9}$$

Equation (2.9) shows that when $(Pd) > 1$, the breakdown potential V_s increases with the increase of (Pd) as the numerator increases linearly whereas due to logarithm term the denominator increases very slowly. On the other side, when $(Pd) < 1$, the breakdown potential V_s increases with the decrease of (Pd) as the numerator decreases linearly whereas due to logarithm term the denominator decreases more rapidly. Thus, V_s is large for both small and large values of (Pd). Obviously, V_s will be minimum for a certain value of (Pd) which can be obtained with the help of the condition

$$\frac{dV_s}{d(Pd)} = 0$$

2.2 Production of plasma through photo-ionization

On absorption of a photon of proper frequency, an atom can go from a lower state to an upper state. The requirement here is that the transition between the two states is radiatively allowed and the energy of the photon is exactly equal to the energy difference between the two states. When energy of a photon is sufficient to move an electron in the atom from a bound state to a free state, ionization of the atom takes place and the liberated electron is free to move outside. If V_i is the ionization

potential of the atom, ionization process may take place when frequency ν of a photon satisfies the condition

$$h\nu \geq eV_i$$

Estimations show that frequency of the required photons for ionization of an atom lies in the ultra-violet, x-rays and γ-rays regions of electromagnetic spectrum. One of the natural examples of production of plasma by photo-ionization is the formation of ionosphere around the earth.

The photons in the visible region of electromagnetic spectrum cannot participate in the ionization process. These photons can excite the atom. The life-time of these excited states is about $10^{-7} - 10^{-8}$ s. Thus, an atom returns back quickly to a lower state. It may lead to a cascading of electron and finally the electron comes to the ground state.

2.3 Production of plasma through thermal ionization

Besides the electronic impact and photo-ionization processes discussed in the preceding sections, plasma can be produced by thermal process. In this process, when a gas is heated, the atoms collide so violently against each other that electrons are knocked off and the atoms get ionized. For ionization potential say 5 eV, the required temperature can be estimated as

$$eV = KT \qquad\qquad 1.62 \times 10^{-19} \times 5 = 1.37 \times 10^{-23} T$$

It gives $T = 59124$ K. It shows that the temperature for thermal ionization cannot be obtained easily in a laboratory. Nowadays attempts are being made to obtain thermal ionization in laboratories and some success has been obtained. In the atmospheres of stars such temperatures of the order of $10^5 - 10^6$ K can be easily found. For example, in the solar corona the temperature is about 2×10^6 K. It helps for production of plasma.

Saha in 1920 developed the theory of thermal ionization and showed that the spectra of stars can be explained by assuming that in the stars,

atoms are excited to higher states and ionized by thermal ionization. For deriving an expression, Saha considered ionization of calcium atom

$$\mathrm{Ca} \rightleftarrows \mathrm{Ca}^+ + \mathrm{e} \tag{2.10}$$

and neglected excitation processes. The process of ionization can be considered as reversible. The total change of thermodynamic potential in a reversible process is zero, so that

$$\delta\phi = 0 \tag{2.11}$$

where ϕ is the thermodynamic potential expressed as

$$\phi = U + PV - TS \tag{2.12}$$

Here, U is internal energy, P the pressure, V the volume, T the temperature and S the entropy. From equations (2.10), (2.11) and (2.12), we have

$$\phi_i + \phi_e - \phi_a = 0 \tag{2.13}$$

where ϕ_i, ϕ_e and ϕ_a, respectively represent thermodynamic potentials for ion, electron and atom. It is customary to use an associated function Ψ instead of ϕ expressed as

$$\Psi = -\frac{\phi}{T} = S - \frac{U + PV}{T}$$

With this definition of associated function Ψ, equation (2.13) gives

$$\Psi_i + \Psi_e - \Psi_a = 0 \tag{2.14}$$

The thermodynamical expression Ψ, for example, for Ca atom is

$$\Psi_a = C_{pa} \ln T - R \ln P_a + R \ln\left[\frac{(2\pi M)^{3/2} K^{5/2} g_a}{h^3}\right] - \frac{U_a}{T}$$

Here, the subscript a refers to atom, C_{pa} is the molecular specific heat, g_a the statistical weight for the neutral atom, U_a the energy at the absolute null point for the temperature, M the mass of atom. Using $C_{pa} = 5R/2$, we have

$$\Psi_a = \frac{5}{2} R \ln T - R \ln P_a + R \ln\left[\frac{(2\pi M)^{3/2} K^{5/2} g_a}{h^3}\right] - \frac{U_a}{T}$$

Thus,

$$\frac{\Psi_a}{R} = \frac{5}{2} \ln T - \ln P_a + \ln\Big[\frac{(2\pi M)^{3/2} K^{5/2} g_a}{h^3}\Big] - \frac{U_a}{RT} \tag{2.15}$$

Similarly for Ca^+ ion, we have

$$\frac{\Psi_i}{R} = \frac{5}{2} \ln T - \ln P_i + \ln\Big[\frac{(2\pi M)^{3/2} K^{5/2} g_i}{h^3}\Big] - \frac{U_i}{RT} \tag{2.16}$$

Here, we have used that mass of ion is almost equal to that of atom. Now, for electron, we have

$$\frac{\Psi_e}{R} = \frac{5}{2} \ln T - \ln P_e + \ln\Big[\frac{(2\pi m)^{3/2} K^{5/2} g_e}{h^3}\Big] - \frac{U_e}{RT} \tag{2.17}$$

Using equations (2.15), (2.16) and (2.17) in (2.14), we have

$$\ln\Big[\frac{P_i P_e}{P_a}\Big] = \frac{5}{2} \ln T + \ln\Big[\frac{(2\pi m)^{3/2} K^{5/2}}{h^3}\Big] + \ln\Big[\frac{g_i g_e}{g_a}\Big] - \frac{U}{RT} \tag{2.18}$$

where $U = U_i - U_a + U_e$. When we start from neutral atoms and go on heating, the atoms break in ions and electrons. Suppose we start with n atoms per unit volume and let a fraction x be ionized. Then we have

$$P_i = nxKT \qquad P_e = nxKT \qquad P_a = nKT(1-x)$$

The total pressure

$$P = P_a + P_i + P_e = nKT(1+x)$$

Thus, we can write

$$P_i = \frac{x}{1+x} P \qquad P_e = \frac{x}{1+x} P \qquad P_a = \frac{1-x}{1+x} P \tag{2.19}$$

Using equation (2.19) in (2.18), we have

$$\ln\Big[\frac{x^2}{1-x^2} P\Big] = \frac{5}{2} \ln T + \ln\Big[\frac{(2\pi m)^{3/2} K^{5/2}}{h^3}\Big] + \ln\Big[\frac{g_i g_e}{g_a}\Big] - \frac{U}{RT}$$

This equation gives

$$\frac{x^2}{1-x^2} P = \frac{g_i g_e}{g_a} \Big(\frac{2\pi m}{h^2}\Big)^{3/2} (KT)^{5/2} \, e^{-U/RT} \tag{2.20}$$

This is Saha's equation for single ionization. When both single and double ionization occur, Saha's equation is modified. Suppose, x_1 and x_2 represent the degree of ionization for single and double ionization, respectively, we can find

$$\frac{x_2(x_1 + 2x_2)}{x_1(1 + x_1 + 2x_2)} P = \frac{g_i g_e}{g_a} \Big(\frac{2\pi m}{h^2}\Big)^{3/2} (KT)^{5/2} \, e^{-U/RT} \tag{2.21}$$

2.3.1 Application of Saha's ionization equation

In order to calculate percentage ionization of element with the help of Saha's formula under varying conditions of temperature and pressure, we need to know the heat of ionization. Let us consider the case of an atom A and the ionization process represented by A $\rightleftarrows$ $A^+ + e$. Suppose V_i eV is the energy required for each ionization. Now, 1 eV = 1.62 $\times 10^{-19}$ J and 4.18 J = 1 calorie. So, 1 eV = 1.62 $\times 10^{-19}/4.18$ = 3.88 $\times 10^{-20}$ calorie. According Avogadro hypothesis, 1 gram mol has 6.17×10^{23} atoms. So energy required for formation of 1 gram mol is

$$U = 3.88 \times 10^{-20} \times 6.17 \times 10^{23}\ V_i = 23940\ V_i \text{ calories}$$

For Na, we have $V_i = 5.12$ eV. Thus, $U = 23940 \times 5.12 = 122$ K calories

In case of Ca, we have

$$g_e = 2 \qquad g_i = 2 \qquad g_a = 1$$

The ionization formula for Ca is therefore

$$\ln\left[\frac{x^2}{1-x^2}\,P\right] = \frac{5}{2}\ln T + \ln\left[\frac{(2\pi m)^{3/2}K^{5/2}}{h^3}\right] + \ln 4 - \frac{U}{4.573T}$$

Thus, the percentage ionization of an atom depends largely on temperature T and ionization potential U. Hence, the percentage ionization of elements in stars under various conditions of temperature and pressure can be obtained with the help of Saha's formula by using the ionization potential and other spectroscopic data for the elements.

2.4 Ionization by exploding wire method

The method of exploding wire is a practical form used for ionization of gas in a laboratory. The ionization in this method is produced by the high temperature produced due to explosion of the wire. The method has two processes.

(i) A large number of high capacity condensers capable of withstanding high voltage are connected in parallel. This system is known as a *bank of condensers*. This system is connected to a high voltage source,

say of the order of 100 k V. The energy stored in the condensers is $\frac{1}{2}CV^2$. For $C = 100\mu$ F and $V = 100 \times 10^3$ V, the energy stored $= 5 \times 10^5$ J.

(ii) The output of the condensers is connected to a metallic wire which can preferably be placed inside a gas tube. This tube can be evacuated to a certain degree of vacuum. The condensers are then discharged through the wire. Due to high increase of temperature, the wire explodes. This temperature can be estimated as the following. Let m be the mass of wire and s the specific heat. Then

$$ms(T - t) = \frac{5 \times 10^5}{4.18}$$

where t is the initial temperature of gas and T when the wire explodes. If $m = 10$ gm, $s = 0.9$ (for Cu) and initial temperature $t = 30^o$ C then

$$10 \times 0.9(T - 30) = \frac{5 \times 10^5}{4.18} \qquad T = 13320^o \text{ C}$$

Because of this sudden increase of temperature, the gas surrounding the wire undergoes the thermal ionization. Thus, the plasma is produced.

2.5 Plasma production by laser

With the development of laser, we got a new tool for plasma production. A gas, which is normally insulating and transparent to radiation at ordinary intensities, rapidly converts into a highly conducting, self-luminous and hot plasma when it is subjected to radiation from a powerful laser. The production of plasma is associated with the shock wave generation. According to Poynting theorem in electrodynamics, for a laser beam of intensity I (watts/cm^2), the electric field E (volts/cm) associated with the wave is $E = 19.3I^{1/2}$. From analogy of electrical breakdown of gases, we can assume that the electrical breakdown of gases with the laser beam and production of plasma are connected with the electric field. For a laser with the output intensity of 10^{11} watts/cm^2, the associated electric field is estimated to be 7 $\times 10^6$ V/cm. The beams from ruby laser ($\lambda =$ 6943 Å) and neodymium laser ($\lambda =$ 10600 Å) for a flash duration $\tau =$ 100 ns have been used in the production of laser plasmas.

The process of plasma production can be divided into three distinct steps: (i) initiation, (ii) formative growth and the onset of the breakdown, and (iii) plasma formation and generation of shock waves and

their propagation. It is a general practice to assume that breakdown of gas takes place when the electron concentration reaches a value of 10^{13} electrons per cc. After the production of plasma, it remains heated for substantially longer time than the duration of the laser flash. Then the energy of the plasma is dissipated by the processes of recombination, diffusion, radiation and conduction, and finally the local thermodynamic equilibrium is attained in a time of the order of 10^{-5} s.

The process of laser ionization is quite different from the process of photon-ionization in the following manner. In a photon-ionization process, a single photon is used for ionization process. But, in the laser-ionization, a multi-photon absorption is possible. For example, for ionization of helium with ionization potential 24.66 eV, in the photon-ionization process, we need a single photon having energy at least 24.66 eV. But in the laser-ionization, a neodymium photon having energy 1.17 eV can easily ionize helium by absorbing as many as twenty two photons. Suppose V_i is the ionization potential of the gas then in the laser-ionization, an atom would require $V_i/h\nu$ quanta to ionize it.

In an atom, we have discrete energy states. In the photon-ionization process, absorption of a photon can only take place when there is resonance between an allowed state and the photon energy. But in the laser-ionization, existence of virtual states is assumed. According to this assumption, after absorption of a photon of frequency ν, the atom can be in a virtual state with energy $h\nu$ for a time $\Delta t = 1/\nu$. In this virtual state, the atom can absorb another photon of frequency ν. After absorption of second photon, the atom will have energy $2h\nu$ and in this state will stay for a smaller time $1/2\nu$. The life-time of successive virtual states goes on decreasing. With the increase of the intensity of laser beam, an atom would be raised to higher and higher energy states until ionization takes place.

Calculations for multi-photon ionization have been carried out by several authors. For example, Bebb and Gold (1966) obtained a general formula for the transition through intermediate states of an atom and found that the main contribution to the sum over the intermediates states is generally made by one or two terms namely those describing transitions to excited levels, the energy of which is very close to a mul-

tiple of the laser photon energy (quasi resonant transitions).

Keldysh (1965) suggested a slightly different mechanism that under the action of extremely high field of the focused laser, there is finite probability for a bound electron to pass through the coulomb barrier and to rise to a higher energy state in the conduction band, *i.e.*, to a free state. Keldysh derived a formula to describe the probability of a transition from a bound state to the virtual levels of continuous spectrum (no quasi resonant transitions). However, this formula found to give approximately the same value of the threshold flux densities as obtained by Bebb and Gold. However, the frequency dependence is some what different.

If F is the uniform photon flux per cm^2 per sec, the ionization rate ϵ due to multi-photon absorption is found to be

$$\epsilon = A\ F^K \tag{2.22}$$

If I is the intensity of the laser beam per unit area then $F = I/h\nu$. The A defined as the probability of ionization per atom per unit time per unit flux is found to be

$$A = \frac{\sigma_r^K}{\nu^{K-1}(K-1)!} \tag{2.23}$$

where σ_r is defined as $\sigma_r = B/F$. Here, B is the rate of excitation interactions from the r^{th} to $(r+1)^{\text{th}}$ state and K is the number of photons the gas atom has to absorb to get ionized. When a gas of volume V at pressure P containing Pn_0V atoms is illuminated for a time t by a constant uniform photon flux F, the number of electrons and ions created by multi-photon absorption is

$$n(t) = \epsilon P n_0 V t \tag{2.24}$$

where n_0 is Avogadro number 6.022×10^{23}. Using equations (3.1) and (3.2) in (3.3), we have

$$n(t) = \frac{P n_0 V t \sigma_r^K F^K}{\nu^{K-1}(K-1)!}$$

If the condition of breakdown is the release of a certain number n_c of electrons in time t, we get

$$n_c = \frac{P n_0 V t \sigma_r^K F^K}{\nu^{K-1}(K-1)!} \qquad F_{TH} = \left(\frac{\nu}{\sigma_r}\right)\left\{\frac{n_c(K-1)!}{P n_0 V t \nu}\right\}^{1/K}$$

where F_{TH} is the threshold flux density for breakdown. Thus, for the multi-photon absorption process, the threshold flux density F_{TH} should be proportional to $P^{-1/K}$ and $t^{-1/K}$, showing that he dependence of F_{TH} on pressure and time is very small. It requires that the product Pt should be very small of the order of 10^{-7} so that no collision ionization processes involving electron atom interactions are involved. It has been experimentally verified in case of helium, argon and nitrogen for 50 Ps flashes of 6900 Å radiation over a wide range of pressure. The very weak dependence of the threshold flux density on pressure indicates that the ionization is mainly dominated by the process of multi-photon absorption.

2.6 Problems and questions

1. Discuss Townsend theory for collisional ionization. Obtain expression for breakdown potential.

2. Discuss about the production of plasma by photo-ionization.

3. Discuss about the production of plasma by thermal ionization.

4. Distinguish between photo-ionization and laser-ionization of a gas.

5. Write short notes on the following

   (i) Ionization by collision

   (ii) Production of plasma by photo-ionization

   (iii) Production of plasma by thermal ionization

   (iv) Production of plasma by lasers

3

Plasma Diagnostics

State of a plasma is expressed in terms of its parameters such as (i) electron and ion densities, (ii) collision frequencies of electrons with neutral atoms, (iii) electron and ion temperatures. Knowledge of these parameters provides information about the plasma. There are some standard methods which can be used for measurement of these parameters. Further, we not only measure these parameters but also study how they change with discharge current, pressure and external magnetic field. In the present chapter, we shall discuss about some standard plasma diagnostic techniques.

3.1 Single probe method

Electrical probes have been a fundamental tool for determination of some characteristics of plasma. In the single probe method, a small insulated wire or a plate (called a probe) is introduced into the plasma and a potential is applied to the probe with respect to one of the electrodes (Figure 3.1). As the probe potential is increased from a relatively negative values through zero to positive values, the current drawn from the plasma will change as shown in Figure 3.2. When the probe potential is sufficiently negative with respect to plasma, the probe will receive only the positive ions and we get the constant current along AB. This current known as the random ion current, denoted by i_{ri}, is determined by the rate at which the ions arrive the probe due to their random motion in the plasma. Its value can be determined by the kinetic theory and is proportional to positive ion concentration in the plasma. As the probe potential becomes less and less negative and finally positive, electrons also reach the probe in addition to ions and the amount of current changes from B to C and the positive ion current still domi-

nates in this region. At the point C, the current becomes zero as equal number of positive ions and electrons are collected by the probe. The probe potential at C is known as the *floating potential.* With the further increase of probe potential, the number of electrons exceeds the number of ions and consequently the current direction is reversed. The current increases rapidly with the increase of potential because more and more electrons reach the probe. Finally at the point D, the electrons reach the probe at their rate determined by the electron density in the plasma and the current becomes almost constant as shown by DE, though the plate voltage is increased. After E, the current may increase rapidly as shown by EF. It is because very high potential accelerates the electrons to such an extent that they can produce ionization of neutral particles which may be present.

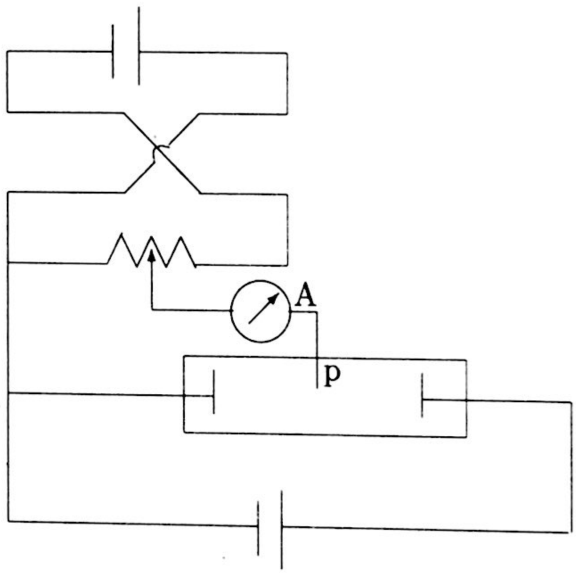


Figure 3.1: Circuit diagram of a single probe method.

3.1.1 Determination of electron temperature T_e

In the linear region CD of Figure 3.2, the number of electrons reaching the probe exceeds the number of positive ions so that the probe may be considered as being surrounded by a sheath in which electrons dominate. Now, only electrons with sufficient energy to overcome this repulsion of the sheath will then penetrate it and reach the probe. Suppose V is the potential difference across the sheath, *i.e.*, the bulk of the plasma and the probe surface, then the electrons having a minimum energy eV

can cross the barrier. Following Boltzmann distribution, the probability that electron will have energy eV is $\exp(-eV/KT_e)$, where T_e is the electron temperature. If n_e is the electron density in the bulk of the plasma and n_p the density in the probe surface, we have

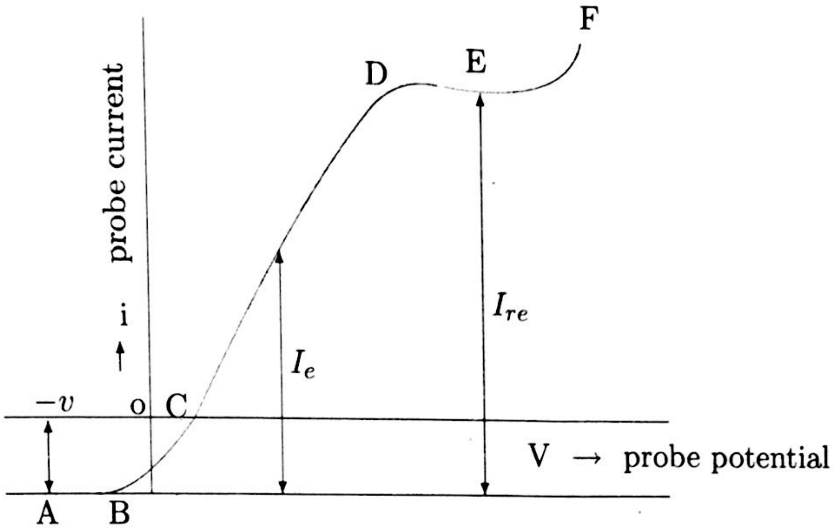


Figure 3.2: Variation of probe current with the probe potential in the single probe method.

$$n_p = n_e \exp(-eV/KT_e) \tag{3.1}$$

Thus, the current is

$$I_p \propto n_e \exp(-eV/KT_e) \qquad I_p = C \exp(-eV/KT_e)$$

where I_p is the probe current and C a constant. When $V = 0$, we have $I_p = C$. That is, the sheath potential becomes zero by the external potential applied to the probe and thus under this condition all the electrons, irrespective of their velocities, reach the probe. This is evidently the electron saturation current represented by the portion DE of the curve and it can be represented by I_{re}. Thus, we have $I_{re} = C$ and therefore,

$$I_p = I_{re} \exp(-eV/KT_e) \tag{3.2}$$

By knowing I_p and I_{re} from the experimental curve, we can determine T_e. But in actual experiment, we can find that the current never saturates and thus I_{re} cannot be obtained precisely. If V_d represents the potential

at which the current saturates, then we have $V_d - V_p = V$, where V_p is the potential applied on the probe. Then from equation (3.2), we have

$$I_p = I_{re}\ \exp[-e(V_d - V_p)/KT_e]$$

Thus, we have

$$\ln I_p = \ln I_{re} - \frac{eV_d}{KT_e} + \frac{eV_p}{KT_e}$$

$$= \text{constant} + \frac{eV_p}{KT_e}$$

A graph of $\ln I_p$ versus V_p should be a straight line and the slope of curve gives T_e, the electron temperature.

3.1.2 Determination of electron density n_e

As mentioned above, it is difficult to get saturation current experimentally. In an actual experiment it is found that at the point D though the current shows a tendency towards saturation, it nevertheless continues to increase. A useful method is to draw a tangent to the curve and the intersection of this tangent with the straight line portion CD of the curve gives the saturation current I_{re}. When v is the velocity of the particle then the assumption that electrons in the plasma follow the Maxwell Boltzmann distribution. Thus, we have

$$v = \left(\frac{KT_e}{2\pi m}\right)^{1/2}$$

and the number of particle striking unit area of the probe per sec is

$$n_e v = n_e\left(\frac{KT_e}{2\pi m}\right)^{1/2}$$

Thus, the random current is

$$I_{re} = Aen_e v = Aen_e\left(\frac{KT_e}{2\pi m}\right)^{1/2} \tag{3.3}$$

where A is the area of the probe. For the known value of T_e, we can determine n_e from equation (3.3).

The ion saturation current obtained from the portion AB of the curve is expressed as

$$I_{ri} = Aen_i\left(\frac{KT_i}{2\pi M}\right)^{1/2}$$

where T_i is the ion temperature and M the mass of the ion. Assuming $n_e = n_i$, T_i can be obtained. These values obtained are however too large.

3.1.3 Limitations of single probe method

The method of single probe, discussed above is too simple. Detailed study of plasma boundary is found to be necessary. For obtaining reliable results, one of the main criteria to be satisfied is that the dimension of the probe should be as small in comparison with the mean free path of the electron in the gas at the corresponding pressure. Moreover, in the deduction of theory, we have assumed that the electrons obey Maxwell Boltzmann distribution. Any departure from linear behaviour of CD in the curve shows that electrons do not follow Maxwell Boltzmann distribution. Again in the above we assumed that current drawn by the probe is so small that it has no effect on the state of plasma. Thus, dimension of the probe should be as small as possible. Nevertheless the conditions around the probe may be quite different those elsewhere in the plasma. A mobile probe may be used to study spatial variation of electron density and electron temperature. However, we do not expect variation of electron temperature with position, specially at low temperature.

3.2 Double probe method

As discussed above, in case of a single probe method, the probe may draw current from the plasma and the state of the plasma may be perturbed. The introduction of probe itself may cause a change in the values of the plasma parameters, which we want do determine. A substitute of the single probe method is the double probe method in which no current is drawn from the plasma. Figure 3.3 shows the circuit diagram of a double probe method. Potential difference between the probes P_1 and P_2 is supplied by a battery, the plasma consists of electrons and ions act like a conductor of electric current. With help of the commutator and the variable resistance, we can change the magnitude as well as direction of

the applied potential difference between P_1 and P_2. The probe current is measured by the ammeter and the potential difference by the voltmeter.

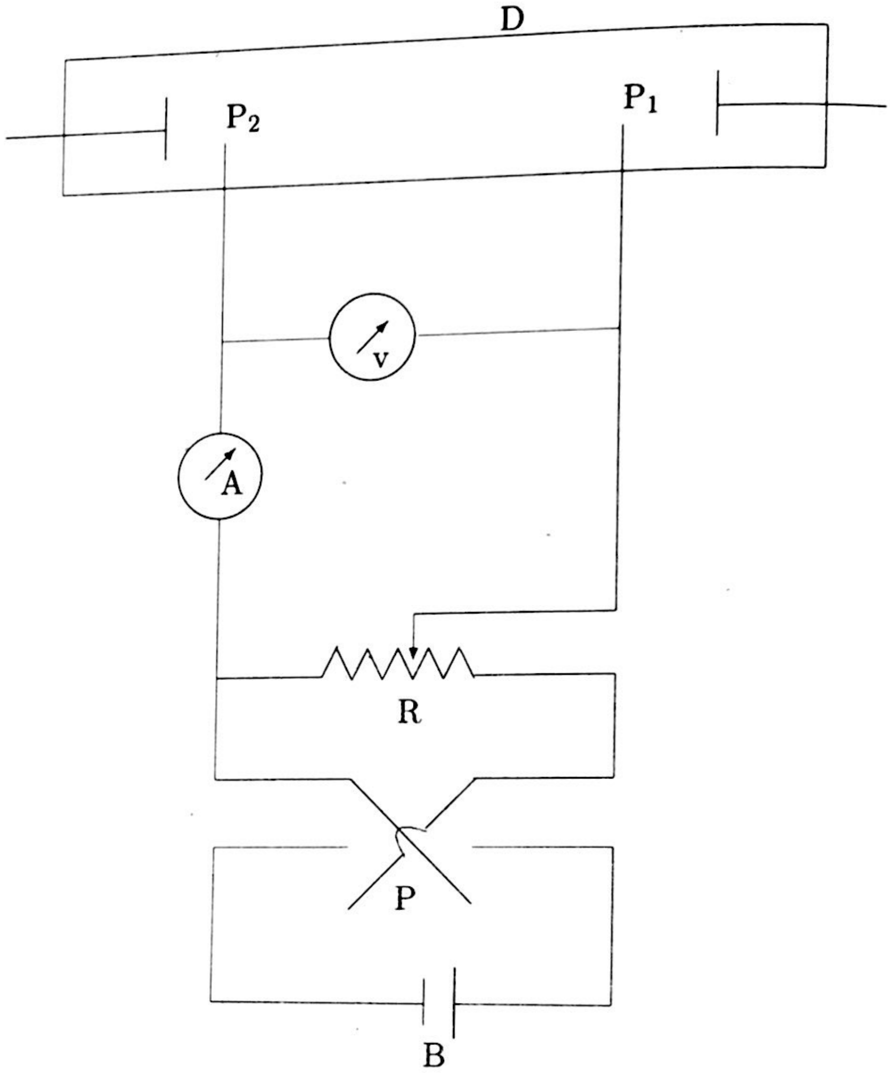


Figure 3.3: Circuit diagram of a double probe method.

An ideal curve showing the variation of probe current with plate voltage is as shown in Figure 3.4. When the P_1 is highly negative, ions are collected by it and the portion AB of the curve is obtained. It represents the ion saturation current. When the voltage of P_1 is made less negative, electrons are also collected by P_1 and the portion BC of the curve is obtained. At the point C, the external voltage between the probes is zero. As the direction of voltage is reversed, P_2 becomes

negative and the portion CDE of the curve is obtained.

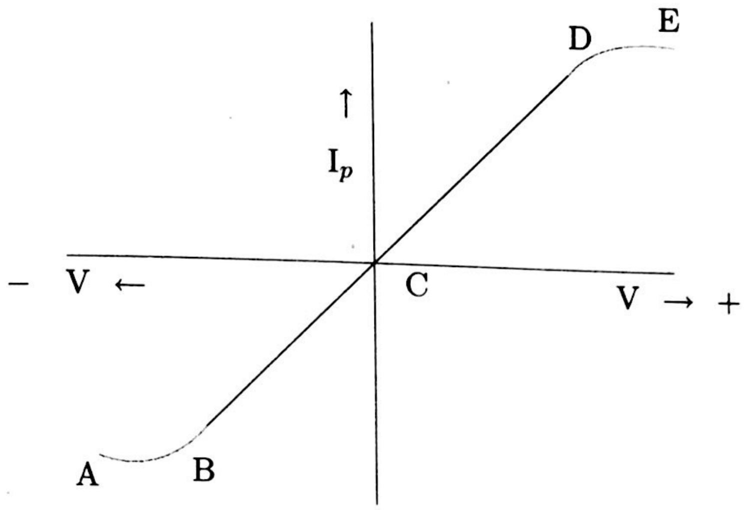


Figure 3.4: Variation of probe current with the probe potential in the double probe method.

As the current at any instant is the difference between the ion saturation current and the electron current, the probe current is

$$I_{ri1} - I_{e2} = \pm I_P \tag{3.4}$$

For the other probe, we have

$$I_{ri2} - I_{e1} = \pm I_P \tag{3.5}$$

From equations (3.4) and (3.5), we have

$$I_{ri1} + I_{ri2} = I_{e1} + I_{e2} \tag{3.6}$$

showing that there is no loss of ions or electrons from the plasma. If I_{re1} and I_{re2} represent the electron saturation current, we have

$$I_{e1} = I_{re1} \exp\Big(-\frac{eV_1}{KT_e}\Big) \qquad I_{e2} = I_{re2} \exp\Big(-\frac{eV_2}{KT_e}\Big) \tag{3.7}$$

From equations (3.6) and (3.7), we have

$$\frac{I_{ri1} + I_{ri2}}{I_{e2}} - 1 = \frac{I_{e1}}{I_{e2}} = \frac{I_{re1}}{I_{re2}} \exp\Big[-\frac{e(V_1 - V_2)}{KT_e}\Big]$$

Thus, we have

$$\ln\Big[\frac{I_{ri1} + I_{ri2}}{I_{e2}} - 1\Big] = \ln\frac{I_{re1}}{I_{re2}} - \frac{e(V_1 - V_2)}{KT_e}$$

$$= \text{constant} - \frac{eV}{KT_e}$$

A plot of $\ln[(I_{ri1}+I_{ri2})/I_{e2}-1]$ versus V should give a straight line. From the slope of this plot, the electron temperature T_e can be determined. From the saturation ion current we have

$$I_{ri} = An_ie\Big(\frac{KT_i}{2\pi M}\Big)^{1/2}$$

By assuming, $n_i = n_e$, the electron density n_e can be determined.

3.3 Magnetic probe

In thermonuclear research, plasma is generally confined with the help of magnetic field in the form cylindrical bottles. The electron density at a distance r from the axis varies when a magnetic field H is applied transversely in the direction of the discharge current and we have

$$n_{eH} = n\ \exp\Big[-\frac{eH_r}{4\sqrt{2mK}}\sqrt{k/T_e}\Big]$$

and the electron temperature is

$$T_{eH} = T_e\Big[1 + C_1\frac{H^2}{P^2}\Big]^{1/2}$$

where H is the magnetic field and $k = 4m/M$ the fraction of energy lost by the electron in collision with a neutral atom, $C_1 = (eL/mv_r)^2$, L the mean free path of electron at a pressure of 1 torr. For a longitudinal magnetic field, the electron temperature is T_{eff} where

$$\frac{T_{eff}}{T_e} = \frac{E_H}{E}$$

and the electron density n_{eH} is

$$\frac{n_{eH}}{n} = \frac{J_0\Big(\frac{r}{\Lambda}\Big[\frac{\nu_{iH}}{\nu_i}\frac{T_e}{T_{eff}}\Big]^{1/2}\Big)}{J_0\Big(\frac{r}{\Lambda}\Big)}$$

where J_0 is the Bessel function of zero order and of the first kind. Here, Λ is the diffusion length, ν_i the collision frequency of electron with neutral atoms. It is obvious that plasma parameters are changed by a magnetic field. For determination of plasma parameters with the probe the following two assumptions are made:

(i) The dimensions of the probe are small as compared to the mean free path of electrons and ions.

(ii) The thickness of space charge sheath surrounding the probe is small as compared to the mean free path of electrons and ions so that the electrons and ions can move in this region undisturbed by collisions.

The question if the Langmuir probe method can be used for determination of electron density and temperature in presence of magnetic field has been investigated by several workers. A magnetic field applied to the plasma effectively reduces the free path of charged particles moving perpendicular to the magnetic field H to less than the radius of curvature $\rho = mv/eH$, where v and m are respectively the velocity and mass of the charged particle. Hence, for a probe collecting across the magnetic field assumption (i) becomes invalid. Thus, for measuring parameters in a magnetic field by probe method, the magnetic field should be small preferably less than 100 G. The assumption (ii) depends on the sheath thickness and thus on the plasma density, the type of gas and on the magnetic field. Therefore, to satisfy the condition (ii), a low density plasma is used and the value of magnetic field is kept low.

Under these assumptions the slope of the portion CD in Figure 3.2 between probe current and probe voltage is used for determination of electron temperature in the same way as in absence of the magnetic field. Hence the electron temperature can be determined from the slope of the curve in presence of magnetic field also. Here, the probe is placed at right angle angles to the direction of the magnetic field.

For large mass of the ion, its gyromagnetic radius is larger than that of electron in the magnetic field. Therefore, the assumption that the mean free path of the ion should be larger than the probe dimension may be valid in case of ions. Thus, the ion density can be determined

from the relation

$$I_{ri} = Aen_i\left(\frac{KT_i}{2\pi M}\right)^{1/2}$$

Considering $n_i = n_e$, the electron density can be determined. Hence, the probe method can also be used for determination of electron density and electron temperature in magnetic field also provided the electron density is small and the magnetic field is low.

3.3.1 Sources of error in probe measurements

(i) The probe method is based on the assumption that the velocity distribution of electrons is Maxwellian. If the distribution is not Maxwellian, the theory of probe becomes invalid.

(ii) The surface of the probes would be incident by the electrons and the energetic electrons may produce secondary electrons from the surface of the probes.

(iii) Energetic photons in the plasma, may produce electrons due to photo emission.

3.4 Microwave method

Because of its small intensity, a microwave signal is of great application for determination of electron density and collision frequency of electrons colliding with neutral particles in a plasma. As the intensity of signal is small, when it is reflected back from plasma or transmitted through the plasma, the state of plasma remains practically undisturbed. Depending on the process (reflection or transmission), there are two methods for determination of plasma parameters with the help of microwaves.

3.4.1 Reflection method

We know that dielectric constant of plasma is

$$\epsilon = 1 - (\omega_p^2/\omega^2)$$

where $\omega_p = \sqrt{ne^2/m\epsilon_0}$ is the plasma frequency and ω the angular frequency of microwave. The refractive index of plasma is

$$\mu = \sqrt{1 - (\omega_p^2/\omega^2)}$$

When the incident frequency $\omega > \omega_p$, the refractive index is real and the wave will transmit through the plasma. On the other side, when $\omega < \omega_p$, the refractive index is imaginary and the wave will be reflected back by the plasma. Thus when the frequency of the microwave signal is gradually increased from $\omega < \omega_p$, then at a certain frequency reflection of wave will stop and the wave will start to transmit through the plasma. The frequency at which the reflection stops and the transmission begins is known as the *critical frequency*. The critical frequency ν_c is expressed as

$$\nu_c = \frac{1}{2\pi}\left(\frac{ne^2}{m\epsilon_0}\right)^{1/2} = 9 \times 10^6\sqrt{n} \text{ Hz}$$

where the density n is expressed in m^{-3}. Klystron oscillators are now easily available and can be used for generation of frequencies of the order of 60 GHz. Hence, electron density of the order of 10^8 m^{-3} can be determined. This principle is the same as generally used for determination of electron density in the ionosphere.

3.4.2 Transmission method

For using the reflection method, one must have a microwave oscillator which could provide a continuous variation of frequency. It is very difficult to have a single oscillator which could provide continuous variation of frequency. Therefore, it is useful to utilize the transmission method. In the transmission method, we have to use a wave of frequency $\omega > \omega_p$. Now, the transmitted wave will undergo both attenuation as well as the phase change. The attenuation will however be small. The experiment is made to measure both the attenuation and the phase shift from which the electron density and collision frequency can be determined.

Let us consider Maxwell equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.8}$$

and

$$\nabla \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t} \tag{3.9}$$

where the current density $\vec{j}$ is due to conduction and $\partial \vec{E}/\partial t$ is the displacement current. Due to collisions of electrons with neutral particles, the energy is lost or absorbed. This energy appears as attenuation.

Expressing $\vec{j}$ as

$$\vec{j} = (\sigma_r - i\sigma_i)\,\vec{E} \tag{3.10}$$

where σ_r and σ_i are real and imaginary parts of conductivity. Using equation (3.10) in (3.9), we have

$$\nabla\times\vec{B} = (\sigma_r - i\sigma_i)\,\vec{E} + \frac{\partial\vec{E}}{\partial t} \tag{3.11}$$

Taking curl of equation (3.8), we have

$$\nabla\times(\nabla\times\vec{E}) = -\frac{\partial}{\partial t}(\nabla\times\vec{B}) \tag{3.12}$$

Using equation (3.11) in (3.12), we have

$$\nabla\times(\nabla\times\vec{E}) = -(\sigma_r - i\sigma_i)\frac{\partial\vec{E}}{\partial t} - \frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla(\nabla.\,\vec{E}) - \nabla^2\,\vec{E} = -(\sigma_r - i\sigma_i)\frac{\partial\vec{E}}{\partial t} - \frac{\partial^2\vec{E}}{\partial t^2}$$

Since there is no accumulation of charge, thus we have $\nabla.\,\vec{E} = 0$ and we have

$$\nabla^2\,\vec{E} = (\sigma_r - i\sigma_i)\frac{\partial\vec{E}}{\partial t} + \frac{\partial^2\vec{E}}{\partial t^2} \tag{3.13}$$

Suppose the direction of propagation of wave is along x-axis and the electric component of microwave field is along y-axis ($i.e.$, the TE mode) then

$$E_y = E_0\, e^{(i\omega t - \gamma x)} \tag{3.14}$$

with $\gamma = \alpha + i\beta$. Here, α is the attenuation per unit length, β is the phase shift per unit length and γ is known as the propagation constant. Now, equation (3.13) becomes

$$\frac{\partial^2 E_y}{\partial x^2} = (\sigma_r - i\sigma_i)\frac{\partial E_y}{\partial t} + \frac{\partial^2 E_y}{\partial t^2} \tag{3.15}$$

Using equation (3.14) in (3.15), we have

$$\gamma^2 = (\sigma_r - i\sigma_i)i\omega - \omega^2 \tag{3.16}$$

Now, we have

$$\sigma_r = \frac{ne^2}{m\epsilon_0}\frac{\omega_c}{\omega_c^2+\omega^2} \qquad \text{and} \qquad \sigma_i = \frac{ne^2}{m\epsilon_0}\frac{\omega}{\omega_c^2+\omega^2} \tag{3.17}$$

Using equation (3.17) in (3.16), we have

$$\gamma^2 = \frac{ne^2}{m\epsilon_0(\omega_c^2+\omega^2)}(\omega_c - i\omega)i\omega - \omega^2$$

Using $\omega_p^2 = ne^2/m\epsilon_0$ here, we have

$$\gamma^2 = \frac{\omega_p^2\omega^2}{(\omega_c^2+\omega^2)} - \omega^2 + i\frac{\omega_p^2\omega_c\omega}{(\omega_c^2+\omega^2)}$$

Thus, we have

$$\alpha^2 - \beta^2 + 2i\alpha\beta = \frac{\omega_p^2\omega^2}{(\omega_c^2+\omega^2)} - \omega^2 + i\frac{\omega_p^2\omega_c\omega}{(\omega_c^2+\omega^2)}$$

It gives

$$\alpha^2 - \beta^2 = \frac{\omega_p^2\omega^2}{(\omega_c^2+\omega^2)} - \omega^2 \qquad \text{and} \qquad 2\alpha\beta = \frac{\omega_p^2\omega_c\omega}{(\omega_c^2+\omega^2)}$$

For microwaves, $(\omega_c/\omega) << 1$ and thus, we have

$$\alpha^2 - \beta^2 = \omega_p^2 - \omega^2 \qquad \text{and} \qquad 2\alpha\beta = \frac{\omega_p^2\omega_c}{\omega} \tag{3.18}$$

Since α is very small, specially for low density plasma, we take $\alpha = 0$ and therefore, we have

$$\beta = \omega\sqrt{1-(\omega_p^2/\omega^2)}$$

Using the value of β in second of equation (3.18), we have

$$\alpha = \frac{\omega_p^2}{2\omega^2}\frac{\omega_c}{\sqrt{1-(\omega_p^2/\omega^2)}}$$

Hence, we see that when $\omega = \omega_p$, we have $\alpha = \infty$ and $\beta = 0$. $\alpha = \infty$ shows that the wave is completely reflected back and $\beta = 0$ shows that there is no propagation. Thus, by measuring the frequency ω_p, electron density n can be determined. Further, by measuring α and ω_p, one can calculate ω_c. When the electron density is uniform then β can be

obtained from the relation $\Delta\phi = (\beta - \beta_0)d$, where d is length of plasma and β_0 is the initial phase difference in absence of plasma. In case of thermonuclear reactions, when the plasma is confined by a magnetic field, the electron density may not be uniform, then we have

$$\Delta\phi = \omega d - \int_0^d \beta \, \mathrm{d}x$$

$$= \omega d - \int_0^d \omega\sqrt{1-(\omega_p^2/\omega^2)}\, \mathrm{d}x = \omega d - \int_0^d \omega\left[1 - \frac{e^2 n(x)}{m\epsilon_0\omega^2}\right]^{1/2} \mathrm{d}x$$

$$= \omega \int_0^d \frac{e^2 n(x)}{2m\epsilon_0\omega^2}\, \mathrm{d}x = \frac{e^2}{2m\epsilon_0\omega}\int_0^d n(x)\, \mathrm{d}x = \frac{e^2 N}{2m\epsilon_0\omega}$$

where N is the column density, *i.e.*, the total number of electrons per unit area in the total path traversed by the microwave.

3.5 Microwave radiation method

Measurements of microwave radiation emitted from a plasma can be used for obtaining electron temperature of the plasma. A plasma, in general, radiates both coherent as well as incoherent radiations. The coherent radiations are considered to arise owing to uncorrelated motion of single particles and can be interpreted statistically in terms of a radiation temperature. Radiations emitted from a hot plasma is not equivalent to the radiations emitted from a black-body. However, there is a narrow frequency range, near the plasma frequency, where the radiations can be approximated to that of a black-body.

For a black-body, the energy density of radiation in thermodynamic equilibrium with matter at temperature T in the frequency range from ν to $\nu + \mathrm{d}\nu$ is expressed by the Planck's law

$$u_\nu \, \mathrm{d}\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT) - 1}\, \mathrm{d}\nu$$

In the microwave region, we have $h\nu << KT$, and the rate of emission of energy from a plasma can be expressed by Rayleigh Jeans equation. The energy radiated in the frequency range from ν to $\nu + \mathrm{d}\nu$ is

$$\mathrm{d}P = \frac{8\pi k T_e \nu^2}{c^3}\, \mathrm{d}\nu$$

where T_e is electron temperature. Radiations can be collected by a microwave antenna. For the effective area A of the antenna (which may be parabolic or a horn antenna) the power collected is

$$A\,\frac{8\pi k T_e \nu^2}{c^3}\,\mathrm{d}\nu$$

It shows that the power P collected will be $CT_e\mathrm{d}\nu$. Here, C is a constant for the mean frequency ν. There may be some uncertainty in the value of C. The power emitted from plasma is compared with a standard noise source whose temperature is known. As the temperature of plasma is much higher than the standard noise source, the intensity of radiations from plasma will be much higher than that of the standard noise source. Therefore, a variable attenuator (whose calibration is known) is used so that the power from the plasma source can be reduced until the two power outputs are equal. The magnitude of attenuation gives the ratio of the electron temperature of plasma and that of the noise source. The essential conditions to be satisfied for getting true temperature of plasma are:

(i) The depth of the medium must be large as compared to the absorption length.

(ii) The antenna must collect all the radiations. Thus, there should not be reflection at the antenna.

This method of measuring electron temperature is based on the assumption that the plasma is radiating like a black-body in the microwave region. Moreover, when the plasma is confined by a magnetic field, due to interaction of magnetic field, the microwave radiations may be emitted and this kind of radiations should be separated out from the true thermal radiations.

3.6 Spectroscopic method

This is a standard method widely used for measurement of temperature of the given source. This method is very accurate in the sense that the plasma is not disturbed at all, as in other methods. The spectral lines emitted by plasma are analyzed by a spectrograph.

3.6.1 Electron temperature from the ratio of spectral line intensities

Let us consider homogeneous plasma in which local thermal equilibrium (LTE) exists. Further, we assume that the plasma is in the steady state and there is no self absorption. Now, the intensity I_{ji} of a line due to transition between the upper level j and lower level i is

$$I_{ji} = n_j \, h\nu_{ji} \, l \, A_{ji} \tag{3.19}$$

where n_j is the population density of upper level of system, ν_{ji} the frequency of spectral line emitted, A_{ji} the spontaneous transition probability, l the thickness of the plasma column along the line-of-sight. According to Maxwell-Boltzmann distribution, we have

$$\frac{n_j}{n_0} = \frac{g_j}{g_0} \exp\left[-\frac{E_j - E_0}{KT_e}\right] \tag{3.20}$$

where n_0 is the population density of the ground level, T_e the electron temperature, and E and g denote the energy and statistical weight of the corresponding states. Using equation (3.20) in (3.19), we have

$$I_{ji} = n_0 h\nu_{ji} \, l \, A_{ji} \frac{g_j}{g_0} \exp\left[-\frac{E_j - E_0}{KT_e}\right] \tag{3.21}$$

Let us consider another transition between the upper state l and lower state k. The intensity of this transition is

$$I_{lk} = n_0 h\nu_{lk} \, l \, A_{lk} \frac{g_l}{g_0} \exp\left[-\frac{E_l - E_0}{KT_e}\right] \tag{3.22}$$

Dividing equation (3.21) by (3.22), we have

$$\frac{I_{ji}}{I_{lk}} = \frac{g_j \nu_{ji} A_{ji}}{g_l \nu_{lk} A_{lk}} \exp\left[\frac{E_l - E_j}{KT_e}\right]$$

Taking logarithm of this equation and rearranging, we have

$$KT_e = (E_l - E_j)/\ln\left[\frac{I_{ji} g_l \nu_{lk} A_{lk}}{I_{lk} g_j \nu_{ji} A_{ji}}\right] \tag{3.23}$$

We now choose two spectral lines and measure the intensities of them. After putting intensities and the known parameters in equation (3.23), we calculate electron temperature of plasma.

In this method we have assumed that local thermal equilibrium exists within the plasma.

3.6.2 Electron temperature from the finite width of spectral lines

It is well known that a finite width of a spectral line is due to the following contributions:

(i) Natural broadening

(ii) Pressure broadening

(iii) Doppler broadening

(iv) Stark broadening

In a gas, natural and pressure broadenings are always present. Doppler broadening is due to motion of particles in a plasma. Stark broadening occurs when a source of spectral line is placed in an external electric field. Even in absence of an external electric field, there is enough electric field in a plasma due to charge separation. Thus, this internal electric field also can cause the Stark broadening. Since this Stark broadening depends on internal field due to ions, this broadening therefore depends on the ion density. Thus, for low density plasma, it is possible to avoid the Stark broadening.

Doppler shift in a gas depends on the distribution of velocities. Further, this distribution must be Maxwellian. The distribution of velocities is thus related to the temperature in the plasma. Hence, when the Doppler broadening in a spectral line can be measured, the temperature of plasma can be determined.

Suppose a particle at rest emits a spectral line of wavelength λ, then for the particle moving with velocity v, the wavelength λ^* for the same transition is

$$\lambda^* = \lambda\Big(1 + \frac{v}{c}\Big)$$

where c is the speed of light. Thus, we have

$$v = c\,\frac{\mathrm{d}\lambda}{\lambda} \tag{3.24}$$

where $\mathrm{d}\lambda = \lambda^* - \lambda$ is the change in wavelength. For the Maxwellian distribution, intensity I_λ of a spectral line is

$$I_\lambda = \text{constant} \times \exp\Big(-\frac{mv^2}{2KT_e}\Big) \tag{3.25}$$

where m is the mass of the emitting ion and T_e the electron density. Using equation (3.25) in (3.24), we have

$$I_\lambda = A \exp\Big(- \frac{mc^2}{2KT_e} \Big[\frac{d\lambda}{\lambda}\Big]^2\Big) \tag{3.26}$$

where A is a constant. For the central intensity of line, we have $d\lambda = 0$ and $I_\lambda = (I_\lambda)_{max}$. Thus, we have $(I_\lambda)_{max} = A$ and therefore,

$$I_\lambda = (I_\lambda)_{max} \exp\Big(- \frac{mc^2}{2KT_e} \Big[\frac{d\lambda}{\lambda}\Big]^2\Big) \tag{3.27}$$

The change in wavelength $d\lambda$ for which the intensity drops to half of the central (maximum) intensity can be obtained as the following:

$$\frac{(I_\lambda)_{max}}{2} = (I_\lambda)_{max} \exp\Big(- \frac{mc^2}{2KT_e} \Big[\frac{d\lambda}{\lambda}\Big]^2\Big)$$

$$\frac{1}{2} = \exp\Big(- \frac{mc^2}{2KT_e} \Big[\frac{d\lambda}{\lambda}\Big]^2\Big)$$

Since the argument of the exponential on right side is small, we can expand it. After neglecting higher order terms, to have

$$\frac{1}{2} = 1 - \frac{mc^2}{2KT_e} \Big[\frac{d\lambda}{\lambda}\Big]^2 \qquad d\lambda = \frac{\lambda}{c}\sqrt{\frac{KT_e}{m}}$$

Then the half-width is

$$\Delta\lambda = \frac{2\lambda}{c}\sqrt{\frac{KT_e}{m}}$$

This relation shows that by measuring the half-width of the spectral line, one can determine electron temperature of the plasma.

3.7 Problems and questions

1. Describe the single probe method for determination of electron temperature T_e and electron density n_e in a plasma. Discuss the sources of error in probe measurements.

2. Describe the double probe method for determination of electron temperature T_e and electron density n_e in a plasma. Discuss the sources of error in probe measurements.

3. Discuss about the probe technique for measurement of plasma parameters in a magnetic field.

4. Discuss about the microwave method for determination of electron density in a plasma.

5. Discuss about the microwave radiation method for determination of electron temperature of plasma.

6. Discuss about the microwave radiation method for determination of electron temperature of plasma.

7. Discuss about the spectroscopic method for determination of electron temperature of plasma.

8. Write short notes on the following

   (i) Single probe method

   (ii) Double probe method

   (iii) Sources of error in probe measurements

   (iv) Microwave radiation method for determination of electron density in a plasma

4

Single Particle Orbit Theory

Plasma can sometimes be considered as a fluid and sometimes as a collection of particles. Here, we are considering a plasma as a collection of particles. In very low density devices like the alternating-gradient synchrotron, collective effects are often unimportant and therefore single particle trajectories may be considered. In this chapter, we shall study behaviour of particles moving in various externally applied fields, such as magnetic, electric and gravitational fields, and combinations of them. We shall assume that the motions of the charged particles do not modify the applied magnetic as well as electric fields in the plasma.

4.1 Particle in a uniform magnetic field

Consider a particle of mass m and charge q moving in a uniform magnetic field $\vec{B}$ with velocity $\vec{v}$. The Lorentz force acting on the particle is $q\,\vec{v} \times \vec{B}$ and the equation of motion of the particle is

$$m\,\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = q\,\vec{v} \times \vec{B} \tag{4.1}$$

Velocity of the particle is non-relativistic and therefore its mass is assumed to remain constant. Here, for convenience, we use the Cartesian coordinate system. Let us consider z-axis of the coordinate system along the direction of the applied uniform magnetic field $\vec{B}$. Thus, $\vec{B} = B\hat{k}$, where B is the magnitude of the field, which is constant. For the velocity

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{4.2}$$

we have

$$\vec{v} \times \vec{B} = v_yB\hat{i} - v_xB\hat{j} \tag{4.3}$$

Using equations (4.2) and (4.3) in (4.1), and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on the two sides of the resultant equation, we get

$$m\,\dot{v_x} = qBv_y \tag{4.4}$$

$$m\,\dot{v_y} = -qBv_x \tag{4.5}$$

$$m\,\dot{v_z} = 0 \tag{4.6}$$

where a dot on a quantity represents its differentiation with respect to the time t. Equation (4.6) shows that z-component of the velocity (*i.e.*, along the direction of the applied magnetic field) is constant.

$$v_z = v_{z0}$$

where v_{z0} is the z-component of initial velocity. Thus, the motion of the particle along the direction of the magnetic field is uniform. Integration of the above equation gives

$$z = v_{z0}t + z_0$$

where z_0 is the constant of integration, showing that the distance along z-axis changes linearly with time provided the particle has initial velocity along the z-axis. On differentiating equation (4.4) with respect to t and using the value of $\dot{v_y}$ from equation (4.5), we get

$$\ddot{v_x} = \frac{qB}{m}\,\dot{v_y} = -\left(\frac{qB}{m}\right)^2 v_x \tag{4.7}$$

Similarly, on differentiating equation (4.5) with respect to t and using the value of $\dot{v_x}$ from equation (4.4), we get

$$\ddot{v_y} = -\frac{qB}{m}\,\dot{v_x} = -\left(\frac{qB}{m}\right)^2 v_y \tag{4.8}$$

Equations (4.7) and (4.8) show that x- and y-components of the velocity have a simple harmonic variation with a frequency, called the cyclotron frequency

$$\omega_c = \frac{|q|\;B}{m}$$

By convention, the cyclotron frequency is always positive, and therefore, we have accounted for the magnitude of the charge. Equation (4.7) can be written as

$$\ddot{v_x} = -\,\omega_c^2\, v_x \tag{4.9}$$

Equation (4.9) is a second order differential equation with constant coefficients and its solution is

$$v_x = v_{x0}\cos(\omega_c t + \phi) \tag{4.10}$$

where v_{x0} is a constant. Using equation (4.10) in (4.4), we have

$$v_y = \frac{m}{qB}\ \dot{v_x} = \pm\frac{1}{\omega_c}\ \dot{v_x} = \mp v_{x0}\sin(\omega_c t + \phi) \tag{4.11}$$

Here, $\pm$ represents the charge of the particle; positive is for a positively charged ion and negative for the negatively charged electron. From equations (4.10) and (4.11), we have

$$v_x^2 + v_y^2 = v_{x0}^2$$

Thus,

$$v_{x0} = \sqrt{v_x^2 + v_y^2} = v_\perp$$

is the velocity of the particle projected in the x-y-plane, denoted by $v_\perp$. Integration of equation (4.10) with respect to t gives

$$x = \frac{v_\perp}{\omega_c}\sin(\omega_c t + \phi) + x_0 \tag{4.12}$$

where x_0 is the constant of integration. Integration of equation (4.11) gives

$$y = \pm\frac{v_\perp}{\omega_c}\cos(\omega_c t + \phi) + y_0 \tag{4.13}$$

where y_0 is the constant of integration. We define the Larmor radius r_L as

$$r_L = \frac{v_\perp}{\omega_c} = \frac{m\ v_\perp}{|q|\ B}$$

so that equations (4.12) and (4.13) can be written as

$$x - x_0 = r_L\sin(\omega_c t + \phi) \tag{4.14}$$

$$y - y_0 = \pm r_L\cos(\omega_c t + \phi) \tag{4.15}$$

Equations (4.14) and (4.15) represent a circle

$$(x - x_0)^2 + (y - y_0)^2 = r_L^2$$

of radius r_L and with the centre at (x_0, y_0). This centre for a charged particle is known as its guiding centre Figure 4.1 shows the motion of two charged particles gyrating about their guiding centers in the plane of the paper. The applied magnetic field is perpendicular to the plane of the paper in the inward direction.

Thus, in the presence of a uniform magnetic field, motion of each charged particle is along a circle, in the plane perpendicular to the direction of the applied magnetic field. The radius of the circle is proportional to the mass and inversely proportional to the charge of the particle. Thus, radius of the orbit of a proton is 1836 times larger than that of the electron.

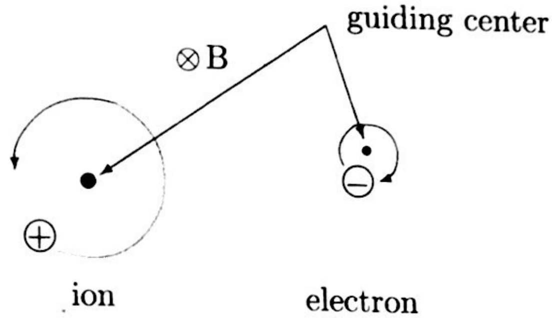


Figure 4.1: Charged particles gyrate about their guiding centers in the plane of the paper when the uniform magnetic field is perpendicular to the plane of the paper in the inward direction. Positive ions and electrons moves in the circular paths in the opposite sense.

A moving charged particle always generates a magnetic field. The direction of motion of the charged particle in the applied magnetic field is such that the direction of the generated magnetic field is opposite to the direction of the applied field. Hence, the positive ions and electrons gyrate in the opposite sense. As the particles of plasma tend to reduce the applied magnetic field, therefore, a plasma is said to be diamagnetic in nature.

In addition to a circular motion in the plane perpendicular to the direction of the applied magnetic field, a particle has an arbitrary velocity v_z along the direction of the applied magnetic field, which remains constant. If a particle has initially a finite value for v_z, the trajectory of the particle is a helix whose axis is along the direction of the applied

field. When $v_z = 0$, the particle moves in a circular orbit about the direction of the applied field.

4.2 Particle in the uniform electric and magnetic fields

Consider a particle of mass m and charge q moving in a uniform magnetic field $\vec{B}$ and a uniform electric field $\vec{E}$ with velocity $\vec{v}$. The Lorentz force acting on the particle is $q[\vec{E} + \vec{v} \times \vec{B}]$ and the equation of motion of the particle is

$$m \frac{d\vec{v}}{dt} = q[\vec{E} + \vec{v} \times \vec{B}] \tag{4.16}$$

Velocity of the particle is non-relativistic and therefore its mass is assumed to remain constant. Here, we use the Cartesian coordinate system. Let us take z-axis of the coordinate system along the direction of the applied uniform magnetic field $\vec{B}$. That is $\vec{B}= B\hat{k}$, where B is a constant magnitude of the field. Now, we take xz-plane so that it contains the applied uniform electric field $\vec{E}$. That is,

$$\vec{E}= E_x\hat{i} + E_z\hat{k} \tag{4.17}$$

where E_x and E_z are constant components of the electric field along x- and z-axes, respectively. For the velocity

$$\vec{v}= v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{4.18}$$

we have

$$\vec{v} \times \vec{B}= v_yB\hat{i} - v_xB\hat{j} \tag{4.19}$$

Using equations (4.17), (4.18) and (4.19) in (4.16), and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on the two sides of the resulting equation, we get

$$m\, \dot{v_x}= qE_x + qBv_y \tag{4.20}$$

$$m\, \dot{v_y}= -qBv_x \tag{4.21}$$

$$m\, \dot{v_z}= qE_z \tag{4.22}$$

where a dot on a quantity represents its differentiation with respect to the time t. Integration of equation (4.22) gives

$$v_z = \frac{qE_z}{m}\, t + v_{z0} \tag{4.23}$$

where v_{z0} is the z-component of the velocity at $t = 0$. Thus, the velocity of the particle along z-direction increases linearly with time t. Further integration of equation (4.23) gives

$$z = \frac{qE_z}{2m}\, t^2 + v_{z0}t + z_0$$

showing that the distance along the z-axis changes as t^2, *i.e*, the particle is accelerated along the z-axis due to the z component of the electric field. Here, z_0 is the constant of integration giving z-component of position at $t = 0$. On differentiating equation (4.20) with respect to t and using the value of $\dot{v_y}$ from equation (4.21), we get

$$\ddot{v_x} = \frac{qB}{m}\, \dot{v_y} = -\left(\frac{qB}{m}\right)^2 v_x \tag{4.24}$$

Similarly, on differentiating equation (4.21) with respect to t and using the value of $\dot{v_x}$ from equation (4.20), we get

$$\ddot{v_y} = -\frac{qB}{m}\, \dot{v_x} = -\frac{qB}{m}\left(\frac{qE_x}{m} + \frac{qB}{m} v_y\right)$$

$$= -\left(\frac{qB}{m}\right)^2 \left(\frac{E_x}{B} + v_y\right) \tag{4.25}$$

On defining the cyclotron frequency $\omega_c = |q|B/m$, equation (4.24) can be written as

$$\ddot{v_x} = -\omega_c^2\, v_x \tag{4.26}$$

Equation (4.26) is a second order differential equation with constant coefficients and its solution is

$$v_x = v_{x0} \cos(\omega_c t + \phi) \tag{4.27}$$

where v_{x0} is constant. Using equation (4.27) in (4.20), we have

$$v_y = \frac{m}{qB}\, \dot{v_x} - \frac{E_x}{B}$$

$$= \pm\frac{1}{\omega_c}\, \dot{v_x} - \frac{E_x}{B} = \mp v_{x0} \sin(\omega_c t + \phi) - \frac{E_x}{B} \tag{4.28}$$

Here, $\pm$ represents the charge of the particle. From equations (4.27) and (4.28), we have

$$v_x^2 + \left(v_y + \frac{E_x}{B}\right)^2 = v_\perp^2$$

so that

$$v_\perp = \sqrt{v_x^2 + \left(v_y + \frac{E_x}{B}\right)^2}$$

Integration of equation (4.27) with respect to t gives

$$x = \frac{v_\perp}{\omega_c}\sin(\omega_c t + \phi) + x_0 \tag{4.29}$$

where x_0 is the constant of integration. Integration of equation (4.28) gives

$$y = \pm\frac{v_\perp}{\omega_c}\cos(\omega_c t + \phi) - \frac{E_x}{B}t + y_0 \tag{4.30}$$

where y_0 is the constant of integration. We define the Larmor radius r_L as

$$r_L = \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B}$$

so that equations (4.29) and (4.30) can be written as

$$x - x_0 = r_L \sin(\omega_c t + \phi) \tag{4.31}$$

$$y - y_0 = \pm r_L \cos(\omega_c t + \phi) - \frac{E_x}{B}\,t \tag{4.32}$$

Equations (4.31) and (4.32) represent a circle

$$\left[x - x_0\right]^2 + \left[y - \left(y_0 - \frac{E_x}{B}\,t\right)\right]^2 = r_L^2$$

of radius r_L and with the centre at $(x_0, y_0 - E_x t/B)$. Hence, y-coordinate of the centre is changing with time in the negative y direction. It shows that the guiding centre drifts with a velocity

$$v_{gc} = \frac{E_x}{B}$$

in the negative y direction. Thus, we have

$$\vec{v}_{gc} = \frac{\vec{E} \times \vec{B}}{B^2} = \vec{v}_E$$

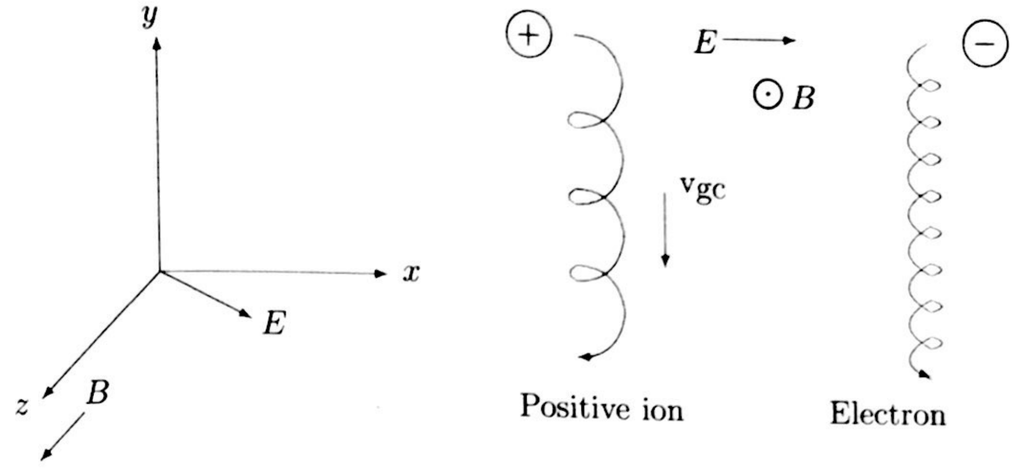


Figure 4.2: Char ed g particles drift in the downward direction (negative y direction) when the applied magnetic field is perpendicular to the plane of the paper in the upward direction.

The drift velocity is independent of mass as well as charge of the particle. Hence, the positive ions as well as electrons drift with the same velocity in the same direction. Figure 4.2 shows drift of an ion and an electron. The drift is in the downward direction (negative y direction) when the applied magnetic field is perpendicular to the plane of the paper in the upward direction.

Physical reason for the drift can be understood in the following manner. The charged particles are gyrating about the direction of magnetic field. For a positive ion, in the first half-cycle when the velocity of the ion has component along the direction of the electric field, the ion gains energy from the electric field and thus, $v_\perp$ increases, and in turn the Larmor radius r_L increases. In the second half-cycle when the velocity of the ion has component opposite to the direction of the electric field, the ion looses energy for opposing the electric field and thus, $v_\perp$ decreases, and in turn r_L decreases. This difference in r_L on the two sides of the orbit causes the drift. An electron gyrates in the opposite direction and also looses and gains energy in the opposite sense, and has a drift. The direction of the drift is the same for the ions as well as for the electrons as it does not depend on the charge and mass of the particle.

Further, the velocity of the particle in the direction of z-axis increases

with time. Actual orbit of the gyrating particle is shown in Figure 4.3.

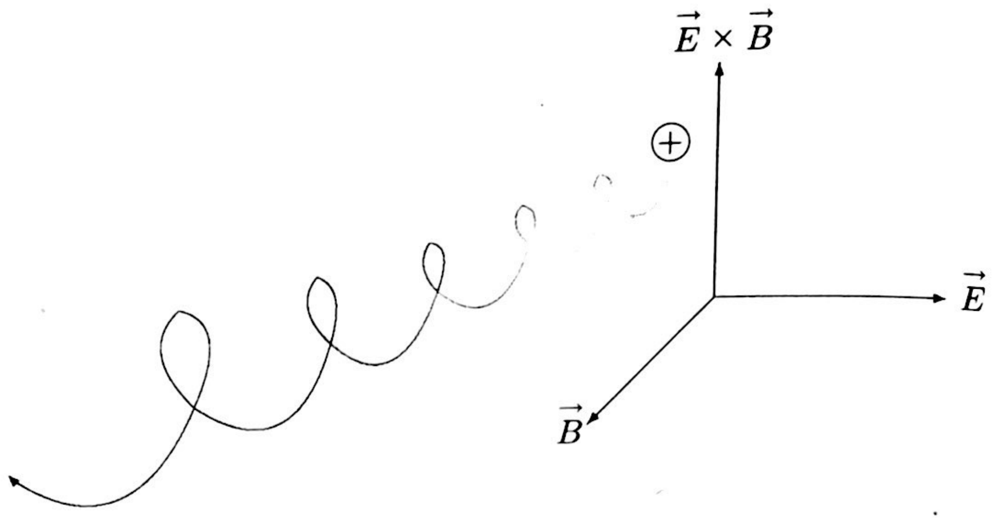


Figure 4.3: Orbit of a gyrating particle in space.

4.3 Particle in a uniform force and a uniform magnetic field

Consider a particle of mass m and charge q moving in a uniform magnetic field $\vec{B}$ and a uniform force $\vec{F}$ with velocity $\vec{v}$. The net force acting on the particle is $\vec{F} + q\, \vec{v} \times \vec{B}$ and the equation of motion of the particle is

$$m \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{F} + q\, \vec{v} \times \vec{B} \tag{4.33}$$

Velocity of the particle is non-relativistic and therefore its mass is assumed to remain constant. Here, we use the Cartesian coordinate system. Let us take z-axis of the coordinate system along the direction of the applied uniform magnetic field $\vec{B}$. That is $\vec{B} = B\hat{k}$, where B is a constant magnitude of the field. Now, we take xz-plane so that it contains the applied uniform force $\vec{F}$. That is,

$$\vec{F} = F_x\hat{i} + F_z\hat{k} \tag{4.34}$$

where F_x and F_z are constant components of the force along x- and z-axes, respectively. For the velocity

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{4.35}$$

we have

$$\vec{v} \times \vec{B} = v_y B\hat{i} - v_x B\hat{j} \tag{4.36}$$

Using equations (4.34), (4.35) and (4.36) in (4.33), and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on the two sides of the resulting equation, we get

$$m\,\dot{v_x} = F_x + qBv_y \tag{4.37}$$

$$m\,\dot{v_y} = -qBv_x \tag{4.38}$$

$$m\,\dot{v_z} = F_z \tag{4.39}$$

where a dot on a quantity represents its differentiation with respect to the time t. Integration of equation (4.39) gives

$$v_z = \frac{F_z}{m}\,t + v_{z0}$$

where v_{z0} is the z-component of the velocity at $t = 0$. Thus, the velocity of the particle along z-direction increases linearly with t. Further integration of this equation gives

$$z = \frac{F_z}{2m}\,t^2 + v_{z0}t + z_0$$

showing that the distance along z-axis changes as t^2, *i.e,* in the particle is accelerated along the z-axis due to the z component of the force. Here, z_0 is the constant of integration giving z-component of position at $t = 0$. On differentiating equation (4.37) with respect to t and using the value of $\dot{v_y}$ from equation (4.38), we get

$$\ddot{v_x} = \frac{qB}{m}\,\dot{v_y} = -\left(\frac{qB}{m}\right)^2 v_x \tag{4.40}$$

Similarly, on differentiating equation (4.38) with respect to t and using the value of $\dot{v_x}$ from equation (4.37), we get

$$\ddot{v_y} = -\frac{qB}{m}\,\dot{v_x} = -\frac{qB}{m}\left(\frac{F_x}{m} + \frac{qB}{m}v_y\right)$$

$$= -\left(\frac{qB}{m}\right)^2\left(\frac{F_x}{qB} + v_y\right) \tag{4.41}$$

On defining the cyclotron frequency $\omega_c = |q|B/m$, equation (4.40) can be written as

$$\ddot{v}_x = -\omega_c^2 v_x \tag{4.42}$$

Equation (4.42) is a second order differential equation with constant coefficients and its solution is

$$v_x = v_{x0} \cos(\omega_c t + \phi) \tag{4.43}$$

where v_{x0} is a constant. Using equation (4.43) in (4.37), we have

$$v_y = \frac{m}{qB}\,\dot{v}_x - \frac{F_x}{qB}$$

$$= \pm\frac{1}{\omega_c}\,\dot{v}_x - \frac{F_x}{qB} = \mp v_{x0}\sin(\omega_c t + \phi) - \frac{F_x}{qB} \tag{4.44}$$

Here, $\pm$ represents the charge of the particle. From equations (4.43) and (4.44), we have

$$v_x^2 + \left(v_y + \frac{F_x}{qB}\right)^2 = v_\perp^2$$

so that

$$v_\perp = \sqrt{v_x^2 + \left(v_y + \frac{F_x}{qB}\right)^2}$$

Integration of equation (4.43) with respect to t gives

$$x = \frac{v_\perp}{\omega_c}\sin(\omega_c t + \phi) + x_0 \tag{4.45}$$

where x_0 is the constant of integration. Integration of equation (4.44) gives

$$y = \mp\frac{v_\perp}{\omega_c}\cos(\omega_c t + \phi) - \frac{F_x t}{qB} + y_0 \tag{4.46}$$

where y_0 is the constant of integration. We define the Larmor radius r_L as

$$r_L = \frac{v_\perp}{\omega_c} = \frac{m v_\perp}{|q|\,B}$$

so that equations (4.45) and (4.46) can be written as

$$x - x_0 = r_L \sin(\omega_c t + \phi) \tag{4.47}$$

$$y - y_0 = \pm r_L \cos(\omega_c t + \phi) - \frac{F_x}{qB}t \tag{4.48}$$

Equations (4.47) and (4.48) represent a circle

$$\left[x - x_0\right]^2 + \left[y - \left(y_0 - \frac{F_x t}{qB}\right)\right]^2 = r_L^2$$

of radius r_L and with the centre at $(x_0,\ y_0 - F_x t/qB)$. Hence, y-coordinate of the centre is changing with time in the negative y direction. It shows that the guiding centre drifts with a velocity

$$v_f = \frac{F_x}{qB}$$

in the negative y direction. Thus, we have

$$\vec{v}_f = \frac{\vec{F} \times \vec{B}}{qB^2}$$

The drift velocity depends on the charge of the particle. Thus, the positive ions and electrons drift in the opposite directions, resulting in a current.

Physical reason for the drift can be understood in the following manner. The charged particles are gyrating about the direction of magnetic field. For a particle, in the first half-cycle when velocity of the particle has component along the direction of the force, the particle gains energy from the force and $v_\perp$ increases, and hence r_L increases. In the second half-cycle when velocity of the particle has component opposite to the direction of the force, the particle loses energy for opposing the force and $v_\perp$ decreases, and hence r_L decreases. This difference in r_L on the two sides of the orbit causes the drift. An electron gyrates in the opposite direction and also loses and gains energy in the opposite sense, and has a drift.

4.3.1 For gravitational force

For the gravitational force $\vec{F} = m\vec{g}$ and the drift velocity is given by

$$\vec{v}_g = \frac{m\vec{g} \times \vec{B}}{qB^2}$$

Here, the drift velocity depends on charge as well as mass of the particle. The oppositely charged particles move in the opposite directions (Figure 4.4), and therefore constitute the current. For the hydrogen plasma of protons and electrons the current density $\vec{j}$ is

$$\vec{j} = n(m_p + m_e)\frac{\vec{g} \times \vec{B}}{B^2}$$

where m_p and m_e are masses of proton and electron, respectively, and n is the density of protons as well as electron.

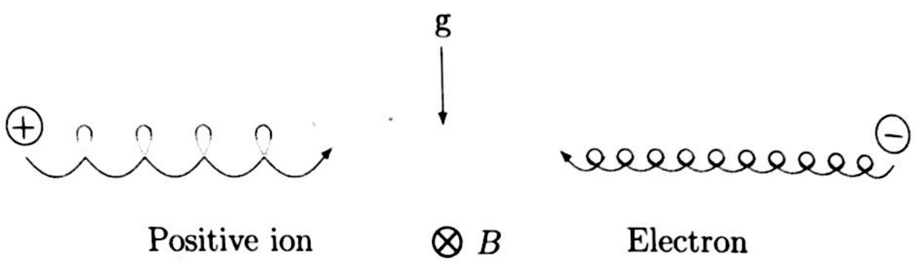


Figure 4.4: Drifts of gyrating particles in a gravitational field. Direction of the applied magnetic field is perpendicular to the plane of the paper in the inward direction. The acceleration due to gravity is in the downward direction.

4.4 Curvature drift

Suppose for a constant magnetic field, the field lines are curve in shape with a constant radius R (Figure 4.5). The particles moving along the field lines experience a centrifugal force

$$\vec{F}_{cf} = \frac{mv_\parallel^2}{R}\hat{r} = \frac{mv_\parallel^2 \vec{R}}{R^2}$$

where m is mass of the particle and $v_\parallel^2$ the average square of the component of velocity of the particle along the magnetic field. The centrifugal force obviously makes an angle with the magnetic field and gives rise to a drift, known as the *curvature drift*, of the guiding centre of the particle with a velocity

$$\vec{v}_R = \frac{1}{q}\frac{\vec{F}_{cf} \times \vec{B}}{B^2}$$

$$= \frac{1}{q}\frac{mv_\parallel^2 \vec{R}}{R^2} \times \frac{\vec{B}}{B^2} = \frac{mv_\parallel^2}{qB^2}\frac{\vec{R} \times \vec{B}}{R^2}$$

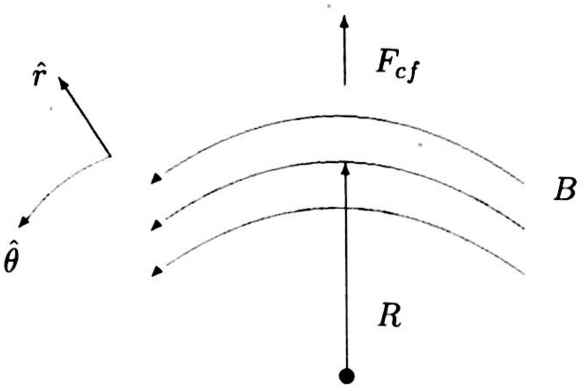


Figure 4.5: Constant magnetic field in the curve shape with a constant radius R.

4.5 Particle in a non-uniform magnetic field

We have discussed about the uniform fields where we could get exact expressions for drifts of the guiding centers. When inhomogeneity in a field is introduced, the situation becomes quite complicated and exact solution is not feasible. Therefore, we go for an approximate solution where we expand in terms of small ratio r_L/L, where r_L is the Larmor radius and L the scale length of the inhomogeneity. This type of theory to obtain approximate solution is known as the *orbit theory*. In the present discussion, we shall discuss about simple cases where only one type of inhomogeneity would be accounted for at a time.

4.5.1 Grad-B drift

For a particle of mass m and charge q moving with velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$, the force $\vec{F}$ acting on the charge is $\vec{F} = q\vec{v} \times \vec{B}$. When z-axis of the Cartesian coordinate system is taken along the direction of the applied magnetic field $\vec{B}$, so that $\vec{B} = B\hat{k}$. Taking $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$, the components of the force are

$$F_x = qBv_y \qquad F_y = -qBv_x \qquad F_z = 0$$

The particle moves in a helical path

$$x = r_L \sin(\omega_c t + \phi) + x_0 \qquad y = \pm r_L \cos(\omega_c t + \phi) + y_0 \qquad z = v_{z0}t + z_0$$

where x_0, y_0 and z_0 are constants and r_L the Larmor radius. The components of the velocity are

$$v_x = v_\perp \cos(\omega_c t + \phi) \qquad v_y = \mp v_\perp \sin(\omega_c t + \phi) \qquad v_z = v_{z0}$$

where v_{z0} is the z-component of initial velocity. The averages[1] of the components of the force due to the field are

$$\langle F_x \rangle = \langle qBv_y \rangle = \langle qB(\mp v_\perp \sin(\omega_c t + \phi)) \rangle$$

$$= \mp qBv_\perp \langle \sin(\omega_c t + \phi) \rangle = 0$$

and

$$\langle F_y \rangle = \langle -qBv_x \rangle = \langle -qBv_\perp \cos(\omega_c t + \phi) \rangle$$

$$= -qBv_\perp \langle \cos(\omega_c t + \phi) \rangle = 0$$

Hence, there is no net force acting on the particle in case of the uniform magnetic field.

Now, we introduce inhomogeneity in the magnetic field by considering that the density of the field lines (also known as the lines of force) increases, say, in the y-direction (Figure 4.6). Hence, the magnetic field

[1] Average values of some trigonometric functions:

$$\langle \sin x \rangle = \frac{\int_0^{2\pi} \sin x \, \mathrm{d}x}{\int_0^{2\pi} \mathrm{d}x} = \frac{[-\cos x]_0^{2\pi}}{2\pi} = 0$$

$$\langle \cos x \rangle = \frac{\int_0^{2\pi} \cos x \, \mathrm{d}x}{\int_0^{2\pi} \mathrm{d}x} = \frac{[\sin x]_0^{2\pi}}{2\pi} = 0$$

$$\langle \sin x \cos x \rangle = \frac{\int_0^{2\pi} \sin x \cos x \, \mathrm{d}x}{\int_0^{2\pi} \mathrm{d}x} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \sin(2x) \, \mathrm{d}x$$

$$= \frac{1}{4\pi} [-\cos(2x)]_0^{2\pi} = 0$$

$$\langle \cos^2 x \rangle = \frac{\int_0^{2\pi} \cos^2 x \, \mathrm{d}x}{\int_0^{2\pi} \mathrm{d}x} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} [1 + \cos(2x)] \, \mathrm{d}x$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \mathrm{d}x + \frac{1}{4\pi} \int_0^{2\pi} \cos(2x) \, \mathrm{d}x = \frac{1}{2}$$

is now a function of y and by using Taylor expansion, we have

$$
\begin{aligned}
B(y) &= B[y_0 \pm r_L \cos(\omega_c t + \phi)] \\
&= B(y_0) \pm r_L \cos(\omega_c t + \phi)\ \frac{\partial B}{\partial y} \\
&= B_0 \pm r_L \cos(\omega_c t + \phi) \frac{\partial B}{\partial y}
\end{aligned}
$$

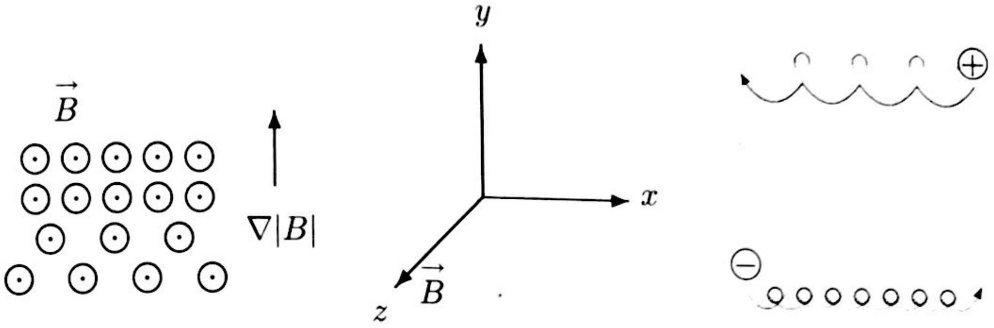


Figure 4.6: Inhomogeneity in the magnetic field is introduced by varying the density of the field lines in the y-direction. The magnetic is directed along the z-axis. The inhomogeneity generates a drift of the guiding centre of the gyrating particle.

Now, we compute averages of the components of the force when the said inhomogeneity is introduced

$$\langle F_x \rangle = \langle qBv_y \rangle = \langle q\Big[B_0 \pm r_L \cos(\omega_c t + \phi)\frac{\partial B}{\partial y}\Big][\mp v_\perp \sin(\omega_c t + \phi)\Big]\rangle$$

$$= \langle \mp qB_0 v_\perp \sin(\omega_c t + \phi)\rangle - \langle q r_L v_\perp \sin(\omega_c t + \phi)\ \cos(\omega_c t + \phi)\frac{\partial B}{\partial y}\rangle$$

$$= \mp qB_0 v_\perp \langle \sin(\omega_c t + \phi)\rangle - q r_L v_\perp \frac{\partial B}{\partial y}\langle \sin(\omega_c t + \phi)\cos(\omega_c t + \phi)\rangle = 0$$

and

$$\langle F_y \rangle = \langle -qBv_x \rangle = \langle -q\Big[B_0 \pm r_L \cos(\omega_c t + \phi)\frac{\partial B}{\partial y}\Big]v_\perp \cos(\omega_c t)\rangle$$

$$= \langle -qB_0 v_\perp \cos(\omega_c t + \phi)\rangle \mp \langle q r_L v_\perp \cos^2(\omega_c t + \phi)\frac{\partial B}{\partial y}\rangle$$

$$= -qB_0 v_\perp \langle \cos(\omega_c t + \phi)\rangle \mp q r_L v_\perp \frac{\partial B}{\partial y}\langle \cos^2(\omega_c t + \phi)\rangle$$

$$= \mp \frac{1}{2} q r_L v_\perp \frac{\partial B}{\partial y}$$

It shows that the average of the x-component of the force is still zero. It is because the inhomogeneity is accounted for in the y-direction. The average of the y-component of the force is non-zero, showing that besides the applied magnetic field in the direction of z-axis, there is a net force $\langle F_y \rangle$ working in the direction of y-axis. This generates drift of the guiding centre with a velocity

$$\vec{v}_{gc} = \frac{\langle F_y \rangle \hat{j} \times B\hat{k}}{qB^2} = \frac{\langle F_y \rangle \hat{i}}{qB}$$

$$= \mp \frac{1}{2} \frac{r_L v_\perp}{B} \frac{\partial B}{\partial y} \hat{i}$$

The choice of y-axis for inhomogeneity of the field is arbitrary. For general variation of magnetic field in space, the drift velocity of the guiding centre due to ∇B is

$$\vec{v}_{\nabla B} = \pm \frac{1}{2} r_L v_\perp \frac{\vec{B} \times \nabla B}{B^2}$$

This drift is known as the *grad-B drift* and is independent of the charge of the particle.

4.5.2 Curvature and grad-B drifts

In the curvature drift, discussed earlier, we accounted for a curved magnetic field with constant magnitude. However, for a curved magnetic field, it is difficult to have a constant magnitude. Hence, for a curved magnetic field, both the curvature drift and grad-B drift arise together. Hence, both of them be considered together. Let us consider a magnetic field (Figure 4.5) where the field lines are along the z-axis of the cylindrical coordinates (r, θ, z). Density of the field lines varies with r so that the field strength decreases with the increase of r. The magnetic field obviously does not depend on θ and z coordinates, and the component B_θ is zero. For a magnetic field $\vec{B}$, we have

$$\nabla . \vec{B} = 0 \qquad\qquad \frac{1}{r} \frac{\partial}{\partial r}(rB_r) = 0$$

Thus,

$$B_r r = \text{constant} = c \qquad\qquad B_r = \frac{c}{r}$$

It shows that the radial component of the magnetic field decreases with the increase of r. Gradient of the magnetic field is

$$\nabla B = \frac{\partial B_r}{\partial r}\hat{r} = -\frac{c}{r^2}\hat{r} = -\frac{B_r}{r}\hat{r}$$

Thus, the grad-B drift velocity is

$$\vec{v}_{\nabla B} = \pm\frac{v_\perp r_L}{2B^2}\,\vec{B}\times\nabla B$$

$$= \pm\frac{v_\perp r_L}{2B^2}\,\vec{B}\times\left(-\frac{B_r}{r}\hat{r}\right)$$

Taking $\vec{B} = B_z\hat{z}$, we have

$$\vec{v}_{\nabla B} = \mp\frac{v_\perp r_L B_r}{2rB_z}\hat{\theta}$$

$$= \mp\frac{v_\perp^2 B_r}{2\omega_c r B_z}\hat{\theta} = \mp\frac{mv_\perp^2 B_r}{2|q|rB_z^2}\hat{\theta} = -\frac{mv_\perp^2 B_r}{2qrB_z^2}\hat{\theta} \tag{4.49}$$

where $v_\perp^2$ is the average square of the velocity-component, perpendicular to the magnetic field, and the Larmor radius r_L and cyclotron frequency ω_c are

$$r_L = \frac{v_\perp}{\omega_c} \qquad\qquad \omega_c = \frac{|q|B}{m}$$

where m and q, respectively, are mass and charge of the particle. The field lines are curve with a radius R (Figure 4.5), and the particles moving along the field lines experience a centrifugal force. Thus, the curvature drift is

$$\vec{v}_R = \frac{mv_\parallel^2}{qB^2}\,\frac{\vec{R}\times\vec{B}}{R^2}$$

where $v_\parallel^2$ is the average square of the velocity-component, along the magnetic field. Using $\vec{B} = B_z\hat{z}$ and $\vec{R} = R\hat{r}$, we have

$$\vec{v}_R = -\frac{mv_\parallel^2}{qB_zR}\hat{\theta} \tag{4.50}$$

Combined drift due to grad-B and curvature of the field lines can be obtained by adding the equations (4.49) and (4.50)

$$\vec{v}_{R+\nabla B}=\vec{v}_R + \vec{v}_{\nabla B}= -\Big(\frac{mv_\perp^2 B_r}{2qrB_z^2}\hat{\theta} + \frac{mv_\parallel^2}{qB_zR}\hat{\theta}\Big) \tag{4.51}$$

From the technology point of view, it is a discouraging situation as both the drifts have the same sign and they add up. For confinement of thermonuclear plasma, when the magnetic field is bent into a torus, the particles would drift out of the torus due to the term $v_\perp$.

For a Maxwellian distribution, average energy of a particle is $3KT/2$. Velocity of a particle can be resolved into two components, parallel and perpendicular to the magnetic field. The parallel component $v_\parallel$ involves one degree of freedom whereas the perpendicular component $v_\perp$ involves two degrees of freedom. Thus,

$$\frac{1}{2}\,m\langle v_\parallel^2\rangle = \frac{1}{2}\,KT \qquad \text{and} \qquad \frac{1}{2}\,m\langle v_\perp^2\rangle = 2\,\frac{1}{2}\,KT$$

$$\langle v_\parallel^2\rangle = \frac{KT}{m} \qquad \text{and} \qquad \langle v_\perp^2\rangle = \frac{2KT}{m} \tag{4.52}$$

Using equations (4.52) in (4.51), we get

$$\begin{aligned}\vec{v}_{R+\nabla B} &= -\Big(\frac{mv_\perp^2 B_r}{2qrB_z^2}\hat{\theta} + \frac{mv_\parallel^2}{qB_zR}\hat{\theta}\Big)\\ &= -\Big(\frac{KT}{m}\frac{mB_r}{qrB_z^2}\hat{\theta} + \frac{KT}{m}\frac{m}{qB_zR}\hat{\theta}\Big)\\ &= -\frac{KT}{qB_z}\Big(\frac{B_r}{rB_z} + \frac{1}{R}\Big)\hat{\theta}\end{aligned}$$

Defining the thermal velocity $v_{th} = \sqrt{2KT/m}$, we have

$$\vec{v}_{R+\nabla B}= -\frac{mv_{th}^2}{2qB_z}\Big(\frac{B_r}{rB_z} + \frac{1}{R}\Big)\hat{\theta} = \mp\frac{v_{th}^2}{2\omega_c}\Big(\frac{B_r}{rB_z} + \frac{1}{R}\Big)\hat{\theta}$$

Here, the $\vec{v}_{R+\nabla B}$ depends on the charge of the particle, but does not depend on its mass.

4.6 Magnetic mirrors

In the preceding section, we discussed about inhomogeneity in the magnetic field where grad-B had an angle with the direction perpendicular to the magnetic field $\vec{B}$. Here, we shall discuss the case when grad-B is along the direction of $\vec{B}$. Let us consider a magnetic field which is primarily pointed in the z-direction of the cylindrical coordinates (r, θ, z) and its magnitude varies in the z-direction (Figure 4.7). The field obviously does not depend on θ, and the component B_θ is zero. Density of the field lines varies radially so that the field strength varies with r. For the magnetic field $\vec{B}$, we have

$$\nabla . \vec{B} = 0 \qquad \frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0 \tag{4.53}$$

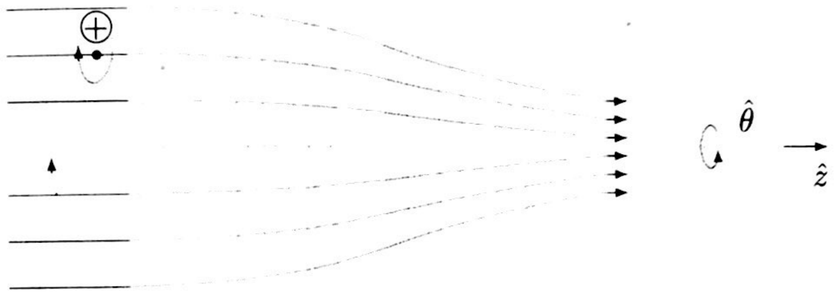


Figure 4.7: Magnetic field is primarily pointed in the z-direction and the magnitude of the field varies with r and z.

Suppose, the variation of the magnetic field is known on the z-axis (*i.e.*, $\partial B_z/\partial z$ is known at $r = 0$), and it does not vary much with r, then equation (4.53) can be integrated as

$$rB_r = -\int_0^r r\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \mathrm{d}r$$

$$= -\left[\frac{\partial B_z}{\partial z}\right]_{r=0}\int_0^r r\ \mathrm{d}r = -\left[\frac{\partial B_z}{\partial z}\right]_{r=0}\frac{r^2}{2}$$

Thus, we have

$$B_r = -\frac{r}{2}\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \tag{4.54}$$

Hence, the variation of the magnetic field with r creates the grad-B

$$\nabla B = \frac{\partial B_r}{\partial r}\ \hat{r} = -\frac{1}{2}\left[\frac{\partial B_z}{\partial z}\right]_{r=0}\hat{r}$$

This ∇B generates drift of the guiding centre with the velocity

$$\vec{v}_{\nabla B} = \pm\frac{1}{2}\, r_L v_\perp \,\frac{\vec{B}\times\nabla B}{B^2}$$

$$= \mp\frac{1}{4}\,\frac{r_L v_\perp}{B}\left[\frac{\partial B_z}{\partial z}\right]_{r=0}\hat{\theta}$$

For the magnetic field $\vec{B}$, velocity $\vec{v}$ and force $\vec{F}\ (= q\vec{v}\times\vec{B})$

$$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} + v_z\hat{z} \qquad \vec{B} = B_r\hat{r} + B_z\hat{z} \qquad \vec{F} = F_r\hat{r} + F_\theta\hat{\theta} + F_z\hat{z}$$

the components of the force are

$$F_r = qv_\theta B_z \qquad F_\theta = qv_z B_r - qv_r B_z \qquad F_z = -qv_\theta B_r$$

The terms $qv_\theta B_z$ and $-qv_r B_z$ give rise to the usual Larmor gyration about the z-direction. The term $qv_z B_r$ vanishes on the z-axis. When it does not vanish, it causes a drift with velocity

$$\vec{v}_f = \frac{1}{q}\,\frac{qv_z B_r\hat{\theta}\times\vec{B}}{B^2} = \frac{v_z B_r}{B}\hat{r}$$

Hence, the drift is in the radial direction and it makes the guiding centers to follow the field lines. The last term $-qv_\theta B_r$ is of great interest in the present context. Using equation (4.54), we have

$$F_z = -qv_\theta B_r = \frac{1}{2}\, qv_\theta r\frac{\partial B_z}{\partial z}$$

Let us now compute average of F_z over one gyration. For simplicity, let us consider a particle whose guiding centre lies on the z-axis. Then because of the symmetry about the z-axis, v_θ is constant during gyration, and depending on the sign of the charge, we have $v_\theta = \mp v_\perp$. Further, for the gyration, r is the Larmor radius r_L. Thus, the average of F_z is

$$\langle F_z\rangle = \frac{1}{2}\, q(\mp v_\perp) r_L\frac{\partial B_z}{\partial z} = \mp\frac{1}{2}\, qv_\perp\frac{v_\perp}{\omega_c}\frac{\partial B_z}{\partial z}$$

$$= \mp\frac{1}{2}\, q\frac{v_\perp^2 m}{|q|B}\frac{\partial B_z}{\partial z} = -\frac{1}{2}\,\frac{mv_\perp^2}{B}\frac{\partial B_z}{\partial z}$$

A gyrating particle creates a magnetic field, and we define the magnetic moment[2] of the particle as

$$\mu = \frac{1}{2}\,\frac{mv_\perp^2}{B} \tag{4.55}$$

so that

$$\langle F_z\rangle = -\mu\frac{\partial B_z}{\partial z}$$

The negative sign shows that it is an example of force acting on a diamagnetic particle. This force along the field lines can be written as

$$\vec{F}_\parallel = -\mu\frac{\partial B}{\partial s}\hat{s} = -\mu\nabla_\parallel B \tag{4.56}$$

where s is the length measured along the field line. It is interesting to notice that as the particle moves in the regions of stronger and weaker magnetic field, the magnetic moment μ remains constant, which can be verified as follows. Equation (4.56) can be expressed as

$$m\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\mu\frac{\partial B}{\partial s}$$

Multiplying this equation by $v_\parallel(=\mathrm{d}s/\mathrm{d}t)$, we have

$$mv_\parallel\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = -\mu\frac{\partial B}{\partial s}\frac{\mathrm{d}s}{\mathrm{d}t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\,mv_\parallel^2\Big) = -\mu\,\frac{\mathrm{d}B}{\mathrm{d}t} \tag{4.57}$$

For conservation of energy of the particle, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\,mv_\parallel^2 + \frac{1}{2}\,mv_\perp^2\Big) = 0 \tag{4.58}$$

[2]When a particle of charge $|q|$ makes $\omega_c/2\pi$ rounds per second in an orbit (loop) of radius r_L we have the current I and the area of cross section A as

$$I = |q|\omega_c/2\pi \qquad\qquad A = \pi r_L^2 = \pi v_\perp^2/\omega_c^2$$

The magnetic moment μ of the current loop is

$$\mu = AI = \frac{\pi v_\perp^2}{\omega_c^2}\,\frac{|q|\omega_c}{2\pi} = \frac{1}{2}\,\frac{v_\perp^2|q|}{\omega_c} = \frac{1}{2}\,\frac{mv_\perp^2}{B}$$

This shows that the definition of the magnetic moment is the usual one.

Using equation (4.55) in (4.58), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\, mv_{\parallel}^2 + \mu B\right) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\, mv_{\parallel}^2\right) + \frac{\mathrm{d}}{\mathrm{d}t}(\mu\, B) = 0 \tag{4.59}$$

Using equation (4.57) in (4.59) we have

$$-\mu\,\frac{\mathrm{d}B}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}(\mu B) = 0 \qquad\qquad B\,\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

Since the magnitude of the magnetic field is not zero, we have

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

showing that μ remains constant. As m and μ for a particle are constant, equation (4.55) gives

$$v_{\perp}^2 \propto B$$

Thus, when a particle moves from a weak-field region to a strong-field region, perpendicular component of the velocity $v_{\perp}$ increases fast as it has degree 2. For the conservation of energy, with the increase of $v_{\perp}$, the parallel component of the velocity $v_{\parallel}$ necessarily decreases fast. If B is high enough in the 'throat' of the field, the $v_{\parallel}$ becomes practically zero, and the particle is reflected back towards a weak-field region where $v_{\parallel}$ increases fact. On the other side also there is another 'throat' where magnetic field is again high and the $v_{\parallel}$ becomes practically zero and the particle is reflected back. Thus, the particle moves back and forth between the two throats (called mirrors) of the magnetic field, and is confined between the mirrors. Figure 4·8 shows a simple pair of coils producing nonuniform magnetic field which behaves like two magnetic mirrors between which a plasma can be trapped. Obviously, a large value of $v_{\perp}$ is required for trapping of particles in the magnetic field. This effect works on both ions and electrons.

This method of trapping plasma is however not perfect. For example for the particles with $v_{\perp} = 0$, there is no magnetic moment (equation 4.55) and it would not experience any magnetic force. Further, for example, at the mid-plane ($B = B_0$), the magnetic field is the lowest, and

following equation (4.55), $v_\perp$ is the minimum and therefore $v_\parallel$ is the maximum. The particles can escape from these regions. Let us obtain a criterion for the escape of the particles.

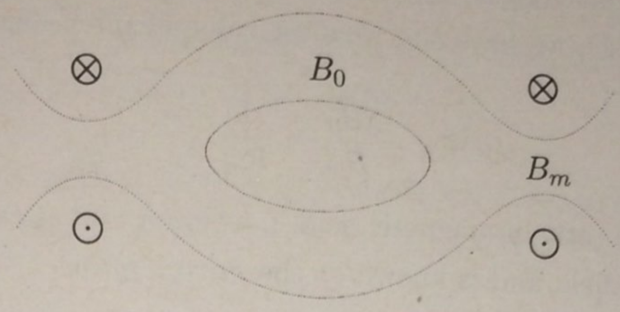


Figure 4.8: A simple pair of coils produces nonuniform magnetic field which behaves like two magnetic mirrors between which a plasma may be trapped.

Suppose at the mid-plane ($B = B_0$) a particle has $v_\perp = v_{\perp 0}$ and $v_\parallel = v_{\parallel 0}$. When this particle reaches a turning point ($B = B_t$), let $v_\perp = v_{\perp t}$ and $v_\parallel = 0$. Then invariance of magnetic moment μ gives

$$\frac{1}{2}\frac{mv_{\perp 0}^2}{B_0} = \frac{1}{2}\frac{mv_{\perp t}^2}{B_t} \tag{4.60}$$

Since in a magnetic field, there is no lose or gain of energy of a particle. Hence, the conservation of energy of the particle gives

$$\frac{1}{2}\,m(v_{\perp 0}^2 + v_{\parallel 0}^2) = \frac{1}{2}\,mv_{\perp t}^2$$

$$v_{\perp t}^2 = v_{\perp 0}^2 + v_{\parallel 0}^2 = v_0^2 \tag{4.61}$$

where v_0 is the magnitude of velocity of the particle. From equation (4.60) and (4.61), we have

$$\frac{B_0}{B_t} = \frac{v_{\perp 0}^2}{v_{\perp t}^2} = \frac{v_{\perp 0}^2}{v_0^2} = \sin^2\theta \tag{4.62}$$

where θ is the pitch angle at the mid-plane. For a particle, smaller value of θ gives smaller value of $v_{\perp 0}$ and in turn smaller value of magnetic moment μ and thus the particle has larger probability of escaping away. These particles which have low value of θ (low value of $v_\perp$) in the weak-field region would have larger value of $v_\perp$ in the high-field region and

will be reflected back. If θ is too small, a high-field region would not have sufficiently high value of $v_\perp$ and therefore no reflection occurs at all.

At a throat replacing B_t by B_m (which is the maximum field strength) in equation (4.62), we have the smallest value of pitch angle θ_m at the mid-plane as

$$\sin^2\theta_m = \frac{B_0}{B_m} = \frac{1}{R_m}$$

where R_m is the ratio of magnetic field in a throat and at the mid-plane of the magnetic field and is known as the *mirror ratio*.

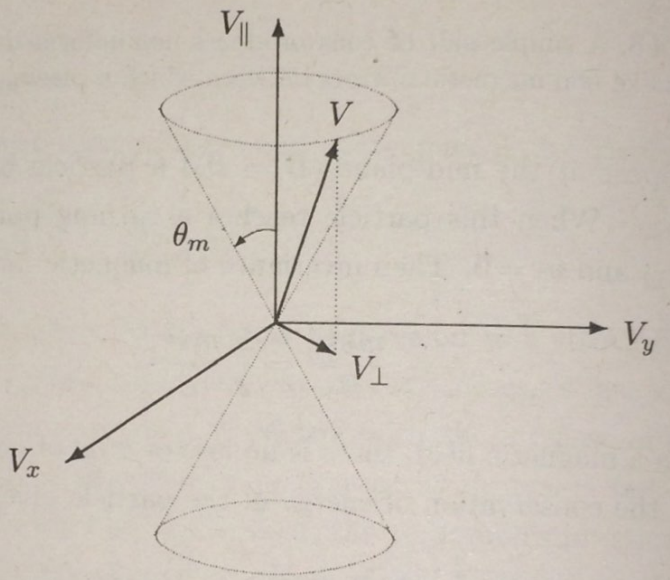


Figure 4.9: Velocity space where θ_m is the minimum value of pitch angle at the mid-plane.

If we consider a velocity space as shown in Figure 4.9, θ_m is the angle of a cone, called the 'loss cone' in the space. This loss cone is independent of mass m and charge q of a particle and depends on the magnetic fields B_0 and B_m. The particles lying within the cone are not confined by the magnetic field and escape out. Electrons are generally lost more easily because they have a higher collision frequency.

The idea of magnetic mirror was first envisaged by Fermi as a mechanism for acceleration of cosmic rays. He proposed that protons bouncing between magnetic mirrors could gain energy in each bounce. Another example of magnetic confinement of plasma are the van Allen belts around

the earth. The magnetic field is strong at the pole and weak at the equator and thus it makes a natural mirror with rather large R_m.

4.7 Particle in a non-uniform electric field and uniform magnetic field

Consider a particle of mass m and charge q moving with a velocity $\vec{v}$ in a uniform magnetic field $\vec{B}$ and a nonuniform electric field $\vec{E}$

$$\vec{E} = E_0 \cos(kx)\hat{i} = E_x\hat{i} \tag{4.63}$$

with velocity $\vec{v}$. Such sinusoidal electric field with wavelength $\lambda = 2\pi/k$ can arise in a plasma during a wave motion through it. The Lorentz force acting on the particle is $q[\vec{E} + \vec{v} \times \vec{B}]$ and the equation of motion of the particle is

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = q[\vec{E} + \vec{v} \times \vec{B}] \tag{4.64}$$

Velocity of the particle is non-relativistic and therefore its mass is assumed to remain constant. Here, we use the Cartesian coordinate system. Let us take z-axis of the coordinate system along the direction of the applied uniform magnetic field $\vec{B}$. That is $\vec{B} = B\hat{k}$, where B is magnitude of the field, which is constant. For the velocity

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{4.65}$$

we have

$$\vec{v} \times \vec{B} = v_yB\hat{i} - v_xB\hat{j} \tag{4.66}$$

Using equations (4.63), (4.65) and (4.66) in (4.64), and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on the two sides of the resultant equation, we get

$$m\,\dot{v_x} = qE_x + qBv_y \tag{4.67}$$

$$m\,\dot{v_y} = -qBv_x \tag{4.68}$$

$$m\,\dot{v_z} = 0 \tag{4.69}$$

where a dot on a quantity represents its differentiation with respect to the time t. Integration of equation (4.69) gives

$$v_z = v_{z0}$$

where v_{z0} is the z-component of the initial velocity. Thus, the motion of the particle along the direction of the magnetic field is uniform. Integration of the above equation gives

$$z = v_{z0}t + z_0$$

where z_0 is the constant of integration. This equation shows that the distance along z-axis changes linearly with time provided the particle has z-component of the initial velocity.

On differentiating equation (4.67) and using the value of $\dot{v}_y$ from equation (4.68), we get (mind that E_x is independent of t)

$$\ddot{v_x} = \frac{qB}{m}\dot{v_y} = -\left(\frac{qB}{m}\right)^2 v_x = -\omega_c^2 v_x \tag{4.70}$$

where $\omega_c = |q|B/m$ is the cyclotron frequency. On differentiating equation (4.68) and using the value of $\dot{v}_y$ from equation (4.67), we get

$$\ddot{v_y} = -\frac{qB}{m}\dot{v_x} = -\frac{q\ B}{m}\left(\frac{qE_0}{m}\cos(kx) + \frac{qB}{m}v_y\right)$$

$$= -\frac{\omega_c^2 E_0}{B}\cos(kx) - \omega_c^2 v_y \tag{4.71}$$

For solving equations (4.70) and (4.71), knowledge of x (*i.e.*, the knowledge of the particle's orbit) is essentially needed. If the electric field is weak, as an approximation, we can use x which was obtained in absence of the electric field (case of uniform magnetic field)

$$x = x_0 + r_L \sin(\omega_c t + \phi) \tag{4.72}$$

Since we are not interested in a gyration at ω_c, it can be taken out by averaging over a cycle. The oscillatory term $\ddot{v_y}$ clearly averages to zero so that

$$\langle\ \ddot{v_y}\ \rangle = 0 = -\frac{\omega_c^2 E_0}{B}\langle \cos\left[k\{x_0 + r_L \sin(\omega_c t + \phi)\}\right]\rangle - \omega_c^2\langle v_y\rangle \tag{4.73}$$

where we have used equation (4.72). Expanding the cosine function, we have

$$\cos\left[k\{x_0 + r_L \sin(\omega_c t + \phi)\}\right] = \cos(kx_0)\cos[kr_L \sin(\omega_c t + \phi)]$$
$$- \sin(kx_0)\sin[kr_L \sin(\omega_c t + \phi)] \tag{4.74}$$

For small Larmor radius, we have $kr_L << 1$ and the Taylor series expansions are

$$\cos[kr_L \sin(\omega_c t + \phi)] = 1 - \frac{1}{2}(kr_L \sin(\omega_c t + \phi))^2 - \ldots$$

$$\sin[kr_L \sin(\omega_c t + \phi)] = kr_L \sin(\omega_c t + \phi) - \ldots$$

after neglecting the higher order terms in the Taylor expansions, and using them in equation (4.74), we have

$$\cos\left[k\{x_0 + r_L \sin(\omega_c t + \phi)\}\right] = \cos(kx_0)[1 - \frac{k^2 r_L^2}{2} \sin^2(\omega_c t + \phi)]$$

$$- \sin(kx_0) kr_L \sin(\omega_c t + \phi)$$

thus,

$$\langle \cos\left[k\{x_0 + r_L \sin(\omega_c t + \phi)\}\right]\rangle = \cos(kx_0)\Big[1 - \frac{k^2 r_L^2}{2}\langle \sin^2(\omega_c t + \phi)\rangle\Big]$$

$$- \sin(kx_0) kr_L \Big\langle \sin(\omega_c t + \phi)\Big\rangle = \cos(kx_0)\Big[1 - \frac{1}{4}\, k^2 r_L^2\Big] \qquad (4.75)$$

Using equation (4.75) in (4.73), we have

$$\langle v_y \rangle = -\frac{E_0}{B}\cos(kx_0)\Big[1 - \frac{1}{4}\, k^2 r_L^2\Big] = -\frac{E_x(x_0)}{B}\Big[1 - \frac{1}{4}\, k^2 r_L^2\Big] \qquad (4.76)$$

Let us now equate the average of the oscillatory terms $\ddot{v_x}$ to zero so that

$$\langle\, \ddot{v_x}\,\rangle = 0 = -\omega_c^2 \langle v_x \rangle \qquad\qquad \langle v_x \rangle = 0$$

Thus, the drift is along the y-direction (equation 4.76) and the usual drift, discussed earlier, is modified to

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}\Big[1 - \frac{1}{4}\, k^2 r_L^2\Big]$$

The correction term depends on k^2 and hence on the second derivative[3] of $\vec{E}$. For the assumed $\vec{E}$, the second derivative is always negative. For

[3]For the electric field $\vec{E}$ we have

$$\vec{E} = E_0 \cos(kx)\hat{i} \qquad \frac{\partial \vec{E}}{\partial x} = -kE_0 \sin(kx)\hat{i} \qquad \frac{\partial^2 \vec{E}}{\partial x^2} = -k^2 E_0 \cos(kx)\hat{i} = -k^2 \vec{E}$$

an arbitrary variation of $\vec{E}$, we need only to replace (ik) by ∇, and the drift velocity is

$$\vec{v_E} = \left[1 + \frac{1}{4}\, r_L^2 \nabla^2\right] \frac{\vec{E} \times \vec{B}}{B^2}$$

Figure 4.10 shows drift of a gyrating particle in a nonuniform electric field $E_0 \cos(kx)\hat{i}$ and uniform magnetic field.

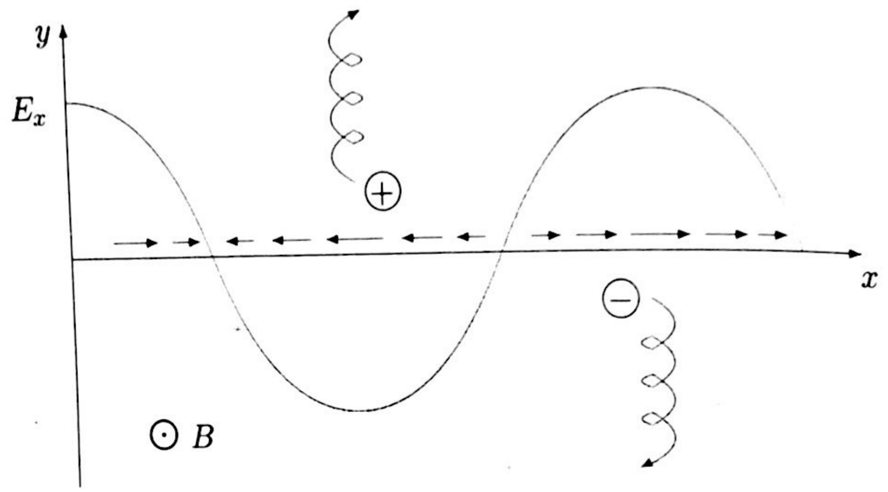


Figure 4.10: Drift of a gyrating particle in a nonuniform electric field $E_0 \cos(kx)\hat{i}$ and uniform magnetic field.

4.8 Particle in a time varying electric field and uniform magnetic field

Consider a particle of mass m and charge q moving in a uniform magnetic field $\vec{B}$ and a time varying electric field $\vec{E}$

$$\vec{E} = E_0 \cos(\omega t + \phi')\hat{i} \tag{4.77}$$

with velocity $\vec{v}$. The Lorentz force acting on the particle is $q[\vec{E} + \vec{v} \times \vec{B}]$ and the equation of motion of the particle is

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = q[\vec{E} + \vec{v} \times \vec{B}] \tag{4.78}$$

Velocity of the particle is non-relativistic and therefore its mass is assumed to remain constant. Here, we use the Cartesian coordinate system. Let us take z-axis of the coordinate system along the direction

of the applied uniform magnetic field $\vec{B}$. That is $\vec{B}= B\hat{k}$, where B is magnitude of the field, which is constant. For the velocity

$$\vec{v}= v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{4.79}$$

we have

$$\vec{v} \times \vec{B}= v_yB\hat{i} - v_xB\hat{j} \tag{4.80}$$

Using equations (4.77), (4.79) and (4.80) in (4.78), and equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on the two sides of the resulting equation, we get

$$m\dot{v_x}= qE_0\cos(\omega t + \phi') + qBv_y \tag{4.81}$$

$$m\dot{v_y}= -qBv_x \tag{4.82}$$

$$m\dot{v_z}= 0 \tag{4.83}$$

where a dot on a quantity represents its differentiation with respect to the time t. Equation (4.83) shows that z-component of the velocity (*i.e.*, along the direction of the applied magnetic field) is constant.

$$v_z = v_{z0}$$

where v_{z0} is the z-component of initial velocity. Thus, the motion of the particle along the direction of the magnetic field is uniform. Integration of the above equation gives

$$z = v_{z0}t + z_0$$

where z_0 is the constant of integration. It shows that the distance along z-axis changes linearly with time provided the particle has z-component of initial velocity. On differentiating equation (4.81) and using the value of $\dot{v_y}$ from equation (4.82), we get

$$\begin{aligned}\ddot{v_x} &= -\frac{\omega qE_0\sin(\omega t + \phi')}{m} + \frac{qB}{m}\dot{v_y} \\ &= -\frac{\omega qE_0\sin(\omega t + \phi')}{m} - \left(\frac{qB}{m}\right)^2 v_x \\ &= \mp\frac{\omega E_0\omega_c\sin(\omega t + \phi')}{B} - \omega_c^2 v_x \end{aligned} \tag{4.84}$$

where $\omega_c = |q|B/m$ is the cyclotron frequency. Similarly, on differentiating equation (4.82) and using the value of $\dot{v_x}$ from equation (4.81), we get

$$\ddot{v_y} = -\frac{qB}{m}\,\dot{v_x}$$

$$= -\frac{qB}{m}\left(\frac{qE_0\cos(\omega t + \phi')}{m} + \frac{qB}{m}v_y\right)$$

$$= -\frac{\omega_c^2 E_0\cos(\omega t + \phi')}{B} - \omega_c^2 v_y \tag{4.85}$$

On defining

$$\overline{v}_P = \mp\frac{\omega E_0\sin(\omega t + \phi')}{\omega_c B} \tag{4.86}$$

$$\overline{v}_E = -\frac{E_0\cos(\omega t + \phi')}{B} \tag{4.87}$$

equations (4.84) and (4.85) can be expressed as

$$\ddot{v_x} = -\omega_c^2(v_x - \overline{v}_P) \tag{4.88}$$

$$\ddot{v_y} = -\omega_c^2(v_y - \overline{v}_E) \tag{4.89}$$

When $\overline{v}_P$ and $\overline{v}_E$ are zero, *i.e.*, there is only uniform magnetic field, the solutions are discussed earlier. Let us try solution of equations (4.88) and (4.89), which is sum of a drift and a gyration, in the following form

$$v_x = v_\perp\cos(\omega_c t + \phi) + \overline{v}_P \qquad\qquad v_y = \mp v_\perp\sin(\omega_c t + \phi) + \overline{v}_E$$

Two times differentiation of equations gives

$$\ddot{v_x} = -\omega_c^2 v_x + (\omega_c^2 - \omega^2)\overline{v}_P \tag{4.90}$$

$$\ddot{v_y} = -\omega_c^2 v_y + (\omega_c^2 - \omega^2)\overline{v}_E \tag{4.91}$$

Equations (4.90) and (4.91) are not the same as (4.88) and (4.89), respectively, unless $\omega^2 << \omega_c^2$. This condition can be obtained if we assume that $\vec{E}$ is so slowly varying such that $\omega^2 << \omega_c^2$. Then equations (4.88) and (4.89), respectively, are solutions of equations (4.84) and (4.85). Equations (4.86) and (4.87) show that the guiding centre drift

has two components $\bar{v_P}$ and $\bar{v_E}$ along x- and y-directions, respectively. The y-component is the usual drift (discussed earlier)

$$\vec{v}_E = -\frac{E_0\cos(\omega t+\phi')}{B}\hat{j} = \frac{\vec{E}\times\vec{B}}{B^2}$$

except that $\vec{v_E}$ now oscillates slowly with a frequency ω. The x-component, along the direction of the electric field $\vec{E}$, known as *polarization drift*, is

$$\vec{v}_P = \mp\frac{\omega E_0\sin(\omega t+\phi')}{\omega_c B}\hat{i} = \pm\frac{1}{\omega_c B}\frac{\mathrm{d}\vec{E}}{\mathrm{d}t} = \frac{m}{qB^2}\frac{\mathrm{d}\vec{E}}{\mathrm{d}t}$$

The polarization drift depends on the charge of the particle. Thus, the opposite charges move in opposite directions, and they constitute a current. For the hydrogen plasma of protons and electrons the current density $\vec{j}$ is

$$\vec{j}_p = ne(v_{ip}-v_{ep}) = \frac{ne}{eB^2}(m_p+m_e)\frac{\mathrm{d}\vec{E}}{\mathrm{d}t} = \frac{\rho}{B^2}\frac{\mathrm{d}\vec{E}}{\mathrm{d}t}$$

where m_p and m_e are masses of proton and electron, respectively, n is the density of protons as well as electron, and ρ the mass density.

4.9 Particle in a time varying magnetic field

Consider a particle of mass m and charge q moving in a time varying magnetic field $\vec{B}$ with velocity $\vec{v}$. As the Lorentz force $\vec{F}$ $(= q\,\vec{v}\times\vec{B})$ due to the magnetic field $\vec{B}$) is perpendicular to the direction of the force, the magnetic field does not do any work on the particle and therefore, it does not change any energy of the particle. However, when $\vec{B}$ changes with time, it has an associated electric field $\vec{E}$ given by the Maxwell's equation

$$\nabla\times\vec{E} = -\dot{\vec{B}} \tag{4.92}$$

where a dot on $\vec{B}$ represents its differentiation with respect to the time t. This produced electric field $\vec{E}$ is perpendicular to the direction of the magnetic field $\vec{B}$, and it can accelerate/decelerate the charged particles. Now, velocity of a particle is also changing and thus, the Lorentz force

is changing. Owing to the electric field, velocity of the particle is almost in the direction perpendicular to the magnetic field. Let

$$\vec{v}_\perp = \frac{d\vec{l}}{dt}$$

be the transverse velocity of the particle (with $v_\parallel$ negligibly small). Here, $\vec{l}$ is he element of the path along a particle trajectory. For the electric field $\vec{E}$ and magnetic field $\vec{B}$, the equation of motion of the particle is

$$m\frac{d\vec{v}}{dt} = q[\vec{E} + \vec{v} \times \vec{B}]$$

We have $\vec{v} = \vec{v}_\perp$, and therefore,

$$m\frac{d\vec{v}_\perp}{dt} = q[\vec{E} + \vec{v}_\perp \times \vec{B}] \tag{4.93}$$

Taking scalar product of $\vec{v}_\perp$ with equation (4.93), we have

$$m\,\vec{v}_\perp \cdot \frac{d\vec{v}_\perp}{dt} = q\,\vec{v}_\perp \cdot [\vec{E} + \vec{v}_\perp \times \vec{B}]$$

$$\frac{d}{dt}\left(\frac{1}{2}m\,v_\perp^2\right) = q\,\vec{v}_\perp \cdot \vec{E} = q\,\vec{E} \cdot \frac{d\vec{l}}{dt}$$

The change in one gyration is obtained by integrating over one period

$$\left[\frac{1}{2}\,mv_\perp^2\right]_{t=0}^{2\pi/\omega_c} = \int_{t=0}^{2\pi/\omega_c} q\,\vec{E} \cdot \frac{d\vec{l}}{dt}\,dt$$

$$\delta\left(\frac{1}{2}\,mv_\perp^2\right) = \int_{t=0}^{2\pi/\omega_c} q\,\vec{E} \cdot \frac{d\vec{l}}{dt}\,dt \tag{4.94}$$

For slow variation of the magnetic field, the time integral in equation (4.94) can be replaced by a line integral

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \oint q\,\vec{E} \cdot d\vec{l} \tag{4.95}$$

Using Stokes' theorem in equation (4.95), we have

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = q\int_S (\nabla \times \vec{E}) \cdot d\vec{S}$$

$$= -q\int_S \dot{\vec{B}} \cdot d\vec{S} \tag{4.96}$$

where we used equation(4.92). Here, $\vec{S}$ is the surface enclosed by the Larmor orbit and is in the direction of the thumb when the fingers of the right hand point in the direction of $\vec{v}$ (right-hand rule). Since a plasma is diamagnetic, we have $\vec{B} \cdot d\vec{S} < 0$ for ions and $\vec{B} \cdot d\vec{S} > 0$ for electrons. Further, for slowing varying field, $\dot{\vec{B}}$ can be taken out in equation (4.96) and we have

$$\delta\Big(\frac{1}{2}mv_\perp^2\Big) = \pm q\ \dot{B}\ \pi r_L^2 = \pm q\pi\ \dot{B}\ \frac{v_\perp^2}{\omega_c^2}$$

$$= \pm q\pi\ \dot{B}\ \frac{v_\perp^2}{\omega_c}\frac{m}{(\pm qB)} = \frac{\frac{1}{2}mv_\perp^2}{B}\frac{2\pi\ \dot{B}}{\omega_c}$$

The quantity $(2\pi\ \dot{B}\ /\omega_c) = (\dot{B}\ /f_c)$ is the change δB during one rotation and thus, we have

$$\delta\Big(\frac{1}{2}mv_\perp^2\Big) = \mu\delta B \tag{4.97}$$

where the magnetic moment μ is defined as

$$\mu = \frac{mv_\perp^2/2}{B} \tag{4.98}$$

Using equation (4.98) in (4.97), we have

$$\delta(\mu B) = \mu\delta B \qquad \mu\delta B + B\delta\mu = \mu\delta B \qquad B\delta\mu = 0$$

As the magnetic field is non-zero, we have

$$\delta\mu = 0 \tag{4.99}$$

Hence, in the slowly varying magnetic field, the magnetic moment is invariant.

As the magnetic field varies in strength, $v_\perp$ (from equation 4.97) and hence the Larmor orbit expands and contacts, and the particles lose and gain transverse energy. This exchange of energy between the particles and the field is described by equation (4.99), that the magnetic moment remains conserved.

Theorem: The magnetic flux through a Larmor orbit is conserved.

Proof: The flux Φ is given by $\Phi = BS$, where $S = \pi r_L^2$ is the area enclosed by Larmor orbit. Thus,

$$\Phi = B\pi r_L^2 = B\pi \frac{v_\perp^2}{\omega_c^2} = B\pi \frac{v_\perp^2 m^2}{q^2\, B^2}$$

$$= \frac{2\pi m}{q^2} \frac{\frac{1}{2}mv_\perp^2}{B} = \frac{2\pi m}{q^2}\mu$$

Since μ is constant, therefore, the flux through a Larmor orbit is constant.

4.10 Problems and questions

1. Derive expressions for cyclotron frequency and Larmor radius for a charged particle moving in a uniform magnetic field. Show that a plasma is diamagnetic.

2. A uniform magnetic field is applied to a plasma having protons and electrons. Show that the charged particles move in helical paths, and the ratio of radii of the helix for a proton and for an electron is equal to the ratio of their masses.

3. Derive an expression for drift velocity for a charged particle moving in the uniform magnetic and electric fields in the space.

4. Derive an expression for drift velocity for a charged particle moving in a force and magnetic field in the space.

5. Hydrogen plasma is placed in a uniform magnetic field and the gravitational field due to earth which are inclined to each other. Obtain an expression for the current density in the plasma.

6. Discuss the principle of plasma confinement in a magnetic field. Obtain criterion for the escape of particles from the field.

7. Derive an expression for drift velocity for a plasma placed in a sinusoidal electric field along with a uniform magnetic field.

8. Derive an expression for polarization drift when a plasma is placed in a time-varying electric field along with a uniform magnetic field.

9. Prove that the magnetic flux through a Larmor orbit is conserved.

10. Show that for a slowly time-varying magnetic field, the magnetic moment of a particle is conserved.

11. Write short notes on the following

    (i) Grad-B drift

    (ii) Curvature drift

    (iii) Magnetic mirrors

    (iv) Polarization drift

5

Fluid Description of Plasma

In the last chapter, we discussed about motions of charged particles in externally applied electric and magnetic fields. However, the situation in a plasma is not so simple, but complicated one. In fact, the electric and magnetic fields are not prescribed but are determined by the positions and motions of the particles themselves. As the moving charged particles generate fields and the fields direct the particles to move in their orbits. Thus, a self-consistent solution for the particle trajectories and the field patterns need to be found. Further, this situation be allowed to vary with time.

Typical density in a plasma density may be of the order of 10^{18} ion-electron pairs per m^3 and each of the particles follows a complicated trajectory. It is, obviously, not possible to discuss plasma's behaviour on the basis of the trajectories for a large number of particles. However, the plasma's behaviour may be discussed with the help of a rather crude model, generally used in fluid mechanics, where we need not to care for the trajectories of individual particles. There we account for the density of the fluid. In this model, a plasma is treated as a fluid consisting of charged particles. In that scenario, we do not talk about the individual charged particles, but the charge and current densities. In an ordinary fluid, the constituent particles go on moving continuously due to frequent collisions among themselves. Though the particles in a plasma do not collide so frequently, it is fortunate to find that fluid mechanics still works. The reason behind the working of this model will be discussed later on.

5.1 Relation of plasma physics and electromagnetic

5.1.1 Maxwell's equations

Maxwell's equations in vacuum are

$$\epsilon_0 \nabla . \vec{E} = \sigma \tag{5.1}$$

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{5.2}$$

$$\nabla . \vec{B} = 0 \tag{5.3}$$

$$\nabla \times \vec{B} = \mu_0 (\vec{j} + \epsilon_0 \dot{\vec{E}}) \tag{5.4}$$

where ϵ_0 and μ_0 are permittivity and permeability, respectively, for the vacuum. These relations for a medium with permittivity ϵ and permeability μ_m can be obtained as

$$\nabla . \vec{D} = \sigma \qquad \nabla \times \vec{E} = -\dot{\vec{B}} \qquad \nabla . \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{j} + \dot{\vec{D}}$$

where $\vec{D}$ and $\vec{H}$ are given by

$$\vec{D} = \epsilon \vec{E} \qquad \vec{H} = \vec{B} \mu_m$$

Here, σ and $\vec{j}$ stand for the free charge and current densities. The bound charge and current densities arising from polarization and magnetization of the medium are accounted for in terms of ϵ and μ_m. Plasma is rather different from the usual mediums, its constituent particles, ions and electrons, are equivalent to the bound charges and currents. In plasma, these charges move in a complicated manner, and it is not possible to account for their effects in ϵ and μ_m. Therefore, in plasma physics, it is customary to use the vacuum equations (5.1) – (5.4), in which σ and $\vec{j}$ include all (both internal and external) charges and currents, respectively.

5.1.2 Classical treatment of plasma like a magnetic material

In a plasma, each gyrating particle has a magnetic moment, and one may think logically to consider the plasma to be a magnetic material with permeability μ_m. (In order to distinguish between the permeability μ_m

and the magnetic moment μ, the suffix m is added to the former one.) For a ferromagnetic material, magnetization $\vec{M}$ per unit volume of a domain is

$$\vec{M} = \frac{1}{V}\sum_i \vec{\mu}_i$$

where V is the volume of the ferromagnetic domain and $\vec{\mu}_i$'s are individual magnetic moments of its constituent particles. This has the same effect as the bound-current density $\vec{j}_b$

$$\vec{j}_b = \nabla \times \vec{M} \tag{5.5}$$

In the vacuum, equation (5.4) can be written as

$$\mu_0^{-1}\nabla \times \vec{B} = \vec{j}_f + \vec{j}_b + \epsilon_0 \dot{\vec{E}} \tag{5.6}$$

where $\vec{j}_f$ is the free-current density. Using equation (5.5) in (5.6), we get

$$\mu_0^{-1}\nabla \times \vec{B} = \vec{j}_f + \nabla \times \vec{M} + \epsilon_0 \dot{\vec{E}}$$

$$\nabla \times [\mu_0^{-1}\vec{B} - \vec{M}] = \vec{j}_f + \epsilon_0 \dot{\vec{E}}$$

$$\nabla \times \vec{H} = \vec{j}_f + \epsilon_0 \dot{\vec{E}}$$

Here, we have defined

$$\vec{H} = \mu_0^{-1}\vec{B} - \vec{M} \tag{5.7}$$

If we assume a linear relation between $\vec{M}$ and $\vec{H}$ as

$$\vec{M} = \chi_m \vec{H}$$

where χ_m is the magnetic susceptibility. Now, equation (5.7) gives

$$\vec{H} = \mu_0^{-1}\vec{B} - \chi_m \vec{H}$$

$$\vec{B} = \mu_0(1 + \chi_m)\vec{H}$$

$$\vec{B} = \mu_m \vec{H} = \frac{\mu_m}{\chi_m}\vec{M} \tag{5.8}$$

It shows that in a magnetic material, magnetic field $\vec{B}$ (or $\vec{H}$) is proportional to the total magnetic moment $\vec{M}$. In a plasma with magnetic field B, each constituent particle has a magnetic moment μ_i

$$\mu_i = \frac{mv_\perp^2}{2B} \propto \frac{1}{B}$$

and thus,

$$M \propto \frac{1}{B}$$

Hence, in a plasma, relation between M and B is not linear as was the case for magnetic materials (equation 5.8). It is therefore not possible to treat plasma as a magnetic medium.

5.1.3 Classical treatment of dielectrics

For a dielectric material, polarization per unit volume $\vec{P}$ is

$$\vec{P} = \frac{1}{V} \sum_i \vec{p}_i$$

where V is the volume of the material, and $\vec{p}_i$'s are the individual electric moments for the electric dipoles. This gives rise to a bound-charge density

$$\sigma_b = -\nabla . \vec{P} \tag{5.9}$$

In the vacuum, equation (5.1) can be written as

$$\epsilon_0 \nabla . \vec{E} = \sigma_f + \sigma_b \tag{5.10}$$

where σ_f is the free-charge density. Using equation (5.9) in (5.10), we get

$$\begin{aligned} \epsilon_0 \nabla . \vec{E} &= \sigma_f - \nabla . \vec{P} \\ \nabla . [\epsilon_0 \ \vec{E} + \vec{P}] &= \sigma_f \\ \nabla . \vec{D} &= \sigma_f \end{aligned}$$

where we defined

$$\vec{D} = \epsilon_0 \ \vec{E} + \vec{P} \tag{5.11}$$

If we assume a linear relation between $\vec{P}$ and $\vec{E}$ as

$$\vec{P} = \epsilon_0 \chi_e \ \vec{E} \tag{5.12}$$

equation (5.11) gives

$$\begin{aligned} \vec{D} &= \epsilon_0 \ \vec{E} + \epsilon_0 \chi_e \ \vec{E} \\ &= \epsilon_0 (1 + \chi_e) \ \vec{E} = \epsilon \ \vec{E} \end{aligned}$$

where $\epsilon = \epsilon_0 (1 + \chi_e)$ is the dielectric constant. There seems no reason why the relation (5.12) cannot be valid in a plasma.

5.2 Equation of motion for fluid

Maxwell's equations (5.1)–(5.4) tell us about the electric field $\vec{E}$ and the magnetic field $\vec{B}$ for the given state of plasma (*i.e.*, for the given charge and current densities). For a self-consistent solution, we must have an equation describing the plasma's response for the known $\vec{E}$ and $\vec{B}$. In the fluid approximation, a plasma is considered as composed of two or more inter-penetrating fluids, one for each of the species in the plasma. The simplest case of a plasma is when it has only one specie of positively charged ions. For example, in a hydrogen plasma, where the positively charged ions are only the protons. Then we have only two fluids, one for the protons and the other for the electrons. Hence, for one kind of ions, we need only two equations of motion, one for the ion-fluid and the second for the electron-fluid. When the plasma is partially ionized, then we need one more equation of motion for the fluid of neutral atoms. The atoms' fluid would interact with the fluids of ions and electrons only through collisions. The ion-fluid and the electron-fluid, however, interact with each other even in absence of collisions, because of the presence of $\vec{E}$ and $\vec{B}$ generated by themselves.

5.2.1 Convective derivative

Consider a particle of mass m and charge q moving in $\vec{E}$ and $\vec{B}$ with velocity $\vec{v}$. The Lorentz force acting on the particle is $q[\vec{E} + \vec{v} \times \vec{B}]$ and the equation of motion of the particle is

$$m\,\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = q[\vec{E} + \vec{v} \times \vec{B}] \tag{5.13}$$

Velocity of the particle is non-relativistic and therefore its mass is assumed to remain constant. For a plasma, it is a good assumption that there are no collisions, and thus, no thermal motions of the particles. Hence, the particles in a fluid-element have no distribution of velocities, like Maxwellian distribution, and therefore, all of them move together. The average velocity $\vec{u}$ of the particles in the element is obviously the same as the velocity $\vec{v}$ of individual particle. The fluid equation is thus obtained by multiplying equation (5.13) by the particle density n and by replacing $\vec{v}$ by $\vec{u}$ as

$$mn\,\frac{d\vec{u}}{dt} = qn[\vec{E} + \vec{u} \times \vec{B}] \tag{5.14}$$

Now, we have

$$\begin{aligned}
\frac{d}{dt}\vec{u}\,(x,y,z,t) &= \frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{u}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{u}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{u}}{\partial z}\frac{dz}{dt} \\
&= \frac{\partial \vec{u}}{\partial t} + u_x\,\frac{\partial \vec{u}}{\partial x} + u_y\,\frac{\partial \vec{u}}{\partial y} + u_z\,\frac{\partial \vec{u}}{\partial z} \\
&= \frac{\partial \vec{u}}{\partial t} + (\vec{u}\,.\nabla)\,\vec{u}
\end{aligned} \tag{5.15}$$

Using equation (5.15) in (5.14) we get

$$mn\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\,.\nabla)\vec{u}\,\right] = qn[\vec{E} + \vec{u} \times \vec{B}] \tag{5.16}$$

5.2.2 Stress tensor

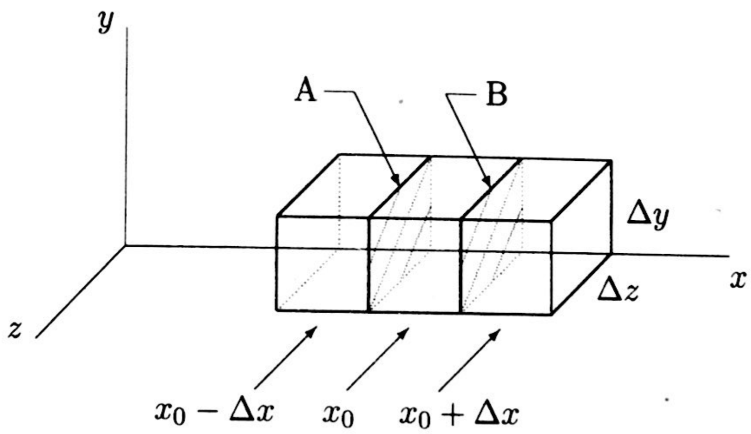


Figure 5. 1: Origin of the elements of stress tensor.

In order to account for the thermal motions, we need to add a term on the right side of equation (5.16) for the pressure force. This force arises from the random motions of the particles in and out of a fluid element. It obviously does not appear in the equation for a single particle as it has no chance to collide with the others. In order to drive an expression for the force, let us first consider only the x-component of the motion.

For that consider a fluid element $\Delta x\ \Delta y\ \Delta z$ centered at $(x_0, \frac{1}{2}\Delta y, \frac{1}{2}\Delta z)$ as shown in Figure 1.

The particles are moving through the faces A and B. The number of particles per second passing through the face A with velocities between v_x and $v_x + \Delta v_x$ is

$$\Delta n_{vx} v_x \Delta y \Delta z$$

where the number density Δn_{vx} of particles moving with velocities between v_x and $v_x + \Delta v_x$ is[1]

$$\Delta n_{vx} = \Delta v_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v_x, v_y, v_z)\ \mathrm{d}v_y\ \mathrm{d}v_z$$

Each particle carries a momentum mv_x. Let n denotes the density and T the temperature at the center of the element. The momentum $P_{\mathrm{A}+}$ carried into the element through A is

$$\begin{aligned} P_{\mathrm{A}+} &= \sum (\Delta n_{vx} v_x\ \Delta y\ \Delta z) m v_x \\ &= \sum \Delta n_{vx} m v_x^2\ \Delta y\ \Delta z = \Delta y\ \Delta z \sum \Delta n_{vx} m v_x^2 \\ &= \Delta y\ \Delta z \sum m v_x^2\ \Delta v_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v_x, v_y, v_z) \mathrm{d}v_y \mathrm{d}v_z \end{aligned}$$

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[1]For Maxwellian distribution function $f(v_x, v_y, v_z)$, number density n is

$$n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v_x, v_y, v_z)\ \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z$$

The number density Δn_{vx} of particles having velocities between v_x and $v_x + \Delta v_x$ is

$$\Delta n_{vx} = \int_{v_x}^{v_x+\Delta v_x}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v_x, v_y, v_z)\ \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z$$

For this small interval from v_x to $v_x + \Delta v_x$, the $f(v_x, v_y, v_z)$ may be considered as independent of the variation of v_x, and therefore,

$$\begin{aligned} \Delta n_{vx} &= \int_{v_x}^{v_x+\Delta v_x} \mathrm{d}v_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v_x, v_y, v_z)\ \mathrm{d}v_y \mathrm{d}v_z \\ &= \Delta v_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v_x, v_y, v_z)\ \mathrm{d}v_y \mathrm{d}v_z \end{aligned}$$

$$= \Delta y\ \Delta z \Big[\int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty m v_x^2 f(v_x, v_y, v_z) \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z \Big]_{\mathrm{A}}$$

$$= \Delta y\ \Delta z \Big[m \langle v_x^2 \rangle \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(v_x, v_y, v_z) \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z \Big]_{\mathrm{A}}$$

$$= \Delta y\ \Delta z \Big[m \langle v_x^2 \rangle \frac{1}{2} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(v_x, v_y, v_z) \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z \Big]_{\mathrm{A}}$$

$$= \Delta y\ \Delta z \Big[m\ \langle v_x^2 \rangle\ \frac{n}{2} \Big]_{\mathrm{A}}$$

where $\langle v_x^2 \rangle$ is the velocity square averaged over the Maxwellian distribution. Similarly, the momentum carried out through the face B is

$$P_{\mathrm{B}+} = \Delta y\ \Delta z \Big[m\ \langle v_x^2 \rangle\ \frac{n}{2} \Big]_{\mathrm{B}}$$

Hence, the net gain in the x-component of momentum for the particles moving in the right direction is

$$P_{\mathrm{A}+} - P_{\mathrm{B}+} = \Delta y\ \Delta z\ m \Big[\Big\{ \frac{n}{2} \langle v_x^2 \rangle \Big\}_{\mathrm{A}} - \Big\{ \frac{n}{2} \langle v_x^2 \rangle \Big\}_{\mathrm{B}} \Big]$$

$$= \Delta y\ \Delta z\ m(-\Delta x) \frac{\partial}{\partial x} \Big(\frac{n}{2} \langle v_x^2 \rangle \Big)$$

$$= -m\ \Delta x\ \Delta y\ \Delta z\ \frac{\partial}{\partial x} \Big(\frac{n}{2}\ \langle v_x^2 \rangle \Big) \tag{5.17}$$

The negative sign comes as we have considered that $\langle v_x^2 \rangle$ at the face A is larger than that at B. In order to include contribution of the particles moving to left direction of the x-axis, equation (5.17) is just doubled. (Since we have the squared term v_x^2, the contributions of the right moving and left moving particles are added.) The total change of momentum of the fluid element at x_0 is

$$\frac{\partial}{\partial t} \Big(\{ nm \Delta x\ \Delta y\ \Delta z \} u_x \Big) = -m\ \Delta x\ \Delta y\ \Delta z\ \frac{\partial}{\partial x} \Big(n\ \langle v_x^2 \rangle \Big)$$

Thus, we have

$$\frac{\partial}{\partial t} \Big(nmu_x \Big) = -m\ \frac{\partial}{\partial x} \Big(n \langle v_x^2 \rangle \Big) \tag{5.18}$$

Let v_x velocity of a particle can be resolved into two components:

$$v_x = u_x + v_{xr} \tag{5.19}$$

where u_x is the x-component of the velocity of plasma-element, and v_{xr} is the random thermal velocity of the particle. Thus, on taking average, we have

$$\langle v_x \rangle = \langle u_x \rangle + \langle v_{xr} \rangle \qquad\qquad \langle v_x \rangle = u_x$$

as the average of the random thermal velocity is zero, and the velocity of the plasma-element is not varying. For one-dimensional Maxwellian distribution, we have

$$\frac{1}{2}\, m\langle v_{xr}^2 \rangle = \frac{1}{2}\, KT \tag{5.20}$$

Using equations (5.19) and (5.20) in (5.18), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\Big(nmu_x\Big) &= -m\frac{\partial}{\partial x}\Big(n\Big\{\langle u_x^2 \rangle + 2\langle u_x v_{xr} \rangle + \langle v_{xr}^2 \rangle\Big\}\Big) \\
&= -m\frac{\partial}{\partial x}\Big(n\Big\{u_x^2 + 2\langle u_x \rangle \langle v_{xr} \rangle + \frac{KT}{m}\Big\}\Big) \\
&= -m\frac{\partial}{\partial x}\Big(n\Big\{u_x^2 + \frac{KT}{m}\Big\}\Big)
\end{aligned} \tag{5.21}$$

Writing nu_x^2 as $(nu_x)(u_x)$ in equation (5.21), we get

$$mn\frac{\partial u_x}{\partial t} + mu_x\frac{\partial n}{\partial t} = -mu_x\frac{\partial}{\partial x}(nu_x) - mnu_x\frac{\partial u_x}{\partial x} - \frac{\partial}{\partial x}(nKT) \tag{5.22}$$

Using the equation of particle conservation[2]

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu_x) = 0$$

in equation (5.22) and defining pressure $p = nKT$, we get

$$mn\frac{\partial u_x}{\partial t} = -mnu_x\frac{\partial u_x}{\partial x} - \frac{\partial p}{\partial x} \qquad\qquad mn\Big(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\Big) = -\frac{\partial p}{\partial x}$$

[2]Equation of continuity is

$$\frac{\partial n}{\partial t} + \nabla.(n\vec{u}) = 0$$

For one-dimensional space, it reduces to

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu_x) = 0$$

This is the usual pressure gradient force and for three-dimensional space can be written as

$$mn\Big[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\,.\nabla)\,\vec{u}\,\Big] = -\nabla p \tag{5.23}$$

After including the electromagnetic forces, we have the fluid equation

$$mn\Big[\frac{\partial \vec{u}}{\partial t} + (\vec{u}\,.\nabla)\,\vec{u}\,\Big] = qn(\vec{E} + \vec{u} \times \vec{B}) - \nabla p \tag{5.24}$$

The term ∇p is due to thermal motion of charged particles. In the derivation of equation (5.23), we assumed that the fluid is isotropic so that the same results as derived for x-direction were used for the y and z directions also. The actual fluid may not be isotropic, and for example, the motion in x direction gives a transfer of momentum in y and z directions. Then in Figure 5.1, the particles may migrate in y and z directions. That is, the number of particles entering at the face A may not be the same as leaving the face B. This shear stress cannot be represented by a scalar p, and the scalar p is replaced by a tensor P. The components $P_{ij} = mn\langle v_i v_j\rangle$ of the stress tensor P provides us information about both the direction of motion and the momentum-component involved. For an anisotropic fluid, ∇p in equations (5.23) and (5.24) is replaced by ∇P, where P is the stress tensor. Let us now consider some simple cases of the stress tensor P.

(i) For the isotropic Maxwellian fluid, the stress tensor P is

$$P = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

so that ∇P is just ∇p.

(ii) We have discussed that in the presence of magnetic magnetic field $\vec{B}$, for a single specie in plasma there may be two temperatures $T_\perp$ and $T_\parallel$ corresponding the motion of charges perpendicular and parallel to the magnetic field, respectively. Thus, there would be two pressures $p_\perp = nKT_\perp$ and $p_\parallel = nkT_\parallel$. For such anisotropic fluid, the stress tensor P is

$$P = \begin{bmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{bmatrix}$$

This tensor also is diagonal, but all the diagonal elements are not equal. However, it still shows isotropy in a plane perpendicular to $\vec{B}$.

In an ordinary fluid, the off-diagonal elements of the stress tensor ∇P are generally non-zero, and they are associated with viscosity of the fluid. When the particles make collisions, they exchange momentum between the colliding partners, and their directions of motion are changed. This process tends to equalize $\vec{u}$ at various points, giving the resistance to shear flow, which is known as the *viscosity*.

5.2.3 Inclusion of collisions

For derivation of equation (5.24), we considered plasma comprising the charged particles. In a partially ionized plasma, the neutral particles exchange momentum with the charged particles through collisions. When a neutral particle moving with velocity $\vec{u}_0$ collides with a charged particle of plasma having velocity $\vec{u}$, the momentum loss per collision will be proportional to the relative velocity $(\vec{u} - \vec{u}_0)$. If the mean free time τ between the collisions is nearly constant, the resulting force term due to the collisions can be written as $-mn(\vec{u} - \vec{u}_0)/\tau$. Hence, to include the collisions, the equation of motion (5.24) (along with anisotropy in the fluid) can be expressed as

$$mn\Big[\frac{\partial \vec{u}}{\partial t} + (\vec{u} .\nabla)\vec{u}\Big] = qn(\vec{E} + \vec{u} \times \vec{B}) - \nabla P - \frac{mn(\vec{u} - \vec{u}_0)}{\tau}$$

Notice that still the collisions between the charged particles have not been included in the discussion.

5.2.4 Comparison with ordinary hydrodynamics

In an ordinary fluid, collisions among its constituent particles are quite frequent, and the fluid obeys the Navier-Stokes equation

$$\rho\Big[\frac{\partial \vec{u}}{\partial t} + (\vec{u} .\nabla)\vec{u}\Big] = -\nabla p + \rho\nu\nabla^2\vec{u} \qquad (5.25)$$

where ρ is the mass density, ν the kinetic viscosity coefficient.

For a plasma, if we neglect collisions between the species (assuming only one specie), equation of motion for a plasma can be expressed as

$$mn\Big[\frac{\partial \vec{u}}{\partial t} + (\vec{u} .\nabla)\vec{u}\Big] = qn(\vec{E} + \vec{u} \times \vec{B}) - \nabla P \qquad (5.26)$$

Here, $mn(=\rho)$ is the mass density. If we take $\rho\nu\nabla^2 \vec{u} = \nabla p - \nabla P$, the equations (5.25) and (5.26) are the same, except the terms having $\vec{E}$ and $\vec{B}$. It is, however, interesting to note that for equation (5.26) the particles do not collide, whereas for the equation (5.25) the collisions among the particles are quite frequent.

If we inquire whether no collisions were accounted for the derivation of the equation (5.26). As we used the relation $\langle v_{xr}^2 \rangle = KT/m$, which is based on the Maxwellian distribution for frequent collisions, it is obvious that the collisions were used implicitly in the derivation. However, the relation was used for taking the average of v_{xr}^2, and any other distribution which gives the same value of v_{xr}^2 can be used here, and the use of Maxwellian distribution is not necessary. The fluid theory of plasmas is therefore not sensitive to deviations from the Maxwellian distribution. The only requirement is that any other distribution should give $\langle v_{xr}^2 \rangle = KT/m$.

Langmuir, in the electrostatic probes, discovered that the electron distribution function was more nearly the Maxwellian than could be obtained through the collisions. This empirical observation, known as the *Langmuir paradox*, has been attributed at times to high frequency collisions. Though no satisfactory resolution to the paradox has been found, but it supports the use of fluid model for plasmas.

Another explanation that the fluid model works for plasmas can be provided in terms of the role played by the magnetic field as that of collisions, in a certain sense. When a charged particle moves in an electric field $\vec{E}$, the field accelerates the particle and its velocity would increase continuously, if allowed. But, there are a large number of particles, and frequent collisions between them are taking place. Maximum velocity of particles come to a limiting value which is proportional to the field $\vec{E}$. When we use magnetic field $\vec{B}$ in addition to $\vec{E}$, the $\vec{B}$ also limits free-streaming of the particles by turning them to gyrate in the Larmor orbits. We also know about the drift velocity $\vec{v}_E$ of charged particles

$$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$$

In this sense, a collision-less plasma, in the presence of $\vec{B}$, behaves like a collisional fluid. Thus, a fluid theory may be taken as valid for plasmas.

5.2.5 Equation of continuity

In a fluid, consider a volume V enclosed by a closed surface S. If the fluid has various species, each of them is conserved separately. Let us consider one specie. Suppose particles are entering into the surface with velocity $\vec{u}$, and at a surface element $\mathrm{d}S$, the particle density is n. Then the particles enter through $\mathrm{d}S$ at the rate $-n\vec{u} \cdot \mathrm{d}\vec{S}$. A negative sign is taken as the directions of $\vec{u}$ and $\mathrm{d}\vec{S}$ are opposite to each other. Thus, the total rate of entering the particles through the closed surface is

$$-\oint n\vec{u} \cdot \mathrm{d}\vec{S} \tag{5.27}$$

With the entering of the particles, the number density in the volume enclosed by the surface increases at the rate $\partial n/\partial t$. Thus, the rate of increase of particles in the volume V is

$$\int_V \frac{\partial n}{\partial t}\, \mathrm{d}V \tag{5.28}$$

Equations (5.27) and (5.28) give the same rate of increase of particles, and therefore,

$$\int_V \frac{\partial n}{\partial t}\, \mathrm{d}V = -\oint n\vec{u} \cdot \mathrm{d}S \tag{5.29}$$

Using divergence theorem, which relates the surface integral to the volume integral, equation (5.29) can be written as

$$\int_V \frac{\partial n}{\partial t}\, \mathrm{d}V = -\int_V \nabla \cdot (n\vec{u})\, \mathrm{d}V \tag{5.30}$$

Since in equation (5.30) both integrals are for the same volume, and therefore their integrands must be equal to give

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n\vec{u}) \qquad\qquad \frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}) = 0 \tag{5.31}$$

This equation is known as the *equation of continuity* for each specie. Any sources or sinks of particles are to be added on the right side of equation (5.31).

5.2.6 Equation of state of plasma

A fluid where the change of energy from the out side is not allowed, the pressure p and density ρ of the fluid are related through the equation of state

$$p = C\rho^{\gamma} \tag{5.32}$$

where C is a constant and γ the ratio of the specific heats C_p/C_v of the fluid. For the particle density n, equation (5.32) can be written as

$$p = C(mn)^{\gamma} = C_1 n^{\gamma} \tag{5.33}$$

where m is mass of a particle, and the $C_1 (= Cm^{\gamma})$ a constant. Thus, we have

$$\nabla p = C_1 \gamma n^{\gamma - 1} \nabla n$$

and therefore,

$$\frac{\nabla p}{p} = \gamma \, \frac{\nabla n}{n} \tag{5.34}$$

For isothermal system, we have $p = nKT$. Thus,

$$\nabla p = \nabla (nKT) = KT \, \nabla n$$

and therefore,

$$\frac{\nabla p}{p} = \frac{\nabla n}{n} \tag{5.35}$$

Comparison of equation (5.35) with (5.34) tells that γ is 1 for an isothermal fluid. For adiabatic compression, T will also change and therefore, the value of γ is larger than 1. For a system with N degrees of freedom, the γ is

$$\gamma = 1 + \frac{2}{N}$$

The validity of the equation of state (equation 5.32) requires the flow of heat to be negligible. Hence, the conductivity of the fluid to be very low. Further, low conductivity is more likely in the direction perpendicular to the magnetic field $\vec{B}$ than parallel to it. However, the basic phenomena are found to be described adequately by the equation of state (5.32).

5.2.7 Complete set of fluid equations

As the simplest case, let us consider a plasma comprising two species: positive ions and electrons. For more species, the extension of the following equations is straight forward. When an ion and an electron in a plasma have charges q_i and q_e, particle densities n_i and n_e, and are moving with velocities $\vec{v_i}$ and $\vec{v_e}$, respectively, the charge density σ and current density $\vec{j}$ are

$$\sigma = n_i q_i + n_e q_e \qquad\qquad \vec{j} = n_i q_i \vec{u_i} + n_e q_e \vec{u_e}$$

In a plasma (fluid) we have a large number of particles. Hence, we shall no more consider motion of a single particle and therefore, we should use $\vec{u}$ for the fluid instead of $\vec{v}$. When we neglect collisions (*i.e.*, fully ionized plasma)[3] and viscosity (*i.e.*, isotropic plasma), equations (5.1) – (5.4), (5.24), (5.31) and (5.32) form the following set of equations:

$$\epsilon_0 \nabla . \vec{E} = n_i q_i + n_e q_e \tag{5.36}$$

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{5.37}$$

$$\nabla . \vec{B} = 0 \tag{5.38}$$

$$\mu_0^{-1} \nabla \times \vec{B} = n_i q_i \vec{u_i} + n_e q_e \vec{u_e} + \epsilon_0 \dot{\vec{E}} \tag{5.39}$$

$$m_j n_j \left[\frac{\partial \vec{u_j}}{\partial t} + (\vec{u_j} . \nabla) \vec{u_j} \right] = q_j \, n_j (\vec{E} + \vec{u_j} \times \vec{B}) - \nabla p_j \qquad j = i, \, e$$

$$\frac{\partial n_j}{\partial t} + \nabla . (n_j \, \vec{u_j}) = 0 \qquad j = i, \, e$$

$$p_j = C_j n_j^{\gamma_j} \qquad j = i, \, e$$

There are 16 scalar unknowns: n_i, n_e, p_i, p_e, $\vec{u_i}$, $\vec{u_e}$, $\vec{E}$ and $\vec{B}$ and 18 scalar equations when we account each vector equation as three scalar equations. Two Maxwell's equations are superfluous as (5.36) and (5.38)

[3] As the collisions between the charged particles and neutral particles are quite frequent whereas the collisions between the charged particles themselves are rare.

can be obtained from the divergence of (5.39) and (5.37).[4] A simultaneous solution of 16 equations in 16 unknowns gives a self-consistent values of fields and motions in the fluid approximation.

5.3 Fluid-drifts perpendicular to $\vec{B}$

A fluid element is composed of a large number of individual particles and the fluid may be expected to have drifts perpendicular to the direction of the magnetic field $\vec{B}$ when individual guiding centers (of the particles) have such drifts. The term ∇p, which appeared in the fluid equations, was not in the equations for the individual particles. Hence, for the fluid there could be one additional drift corresponding to ∇p. For each species in a fluid we have the equation of motion

$$mn\left[\frac{\partial \vec{u}}{\partial t} + (\,\vec{u}\,.\nabla)\,\vec{u}\,\right] = qn(\,\vec{E} + \vec{u} \times \vec{B}\,) - \nabla p \tag{5.40}$$

Let the velocity be written as

$$\vec{u} = (u_\perp \hat{e_\perp} + u_\parallel \hat{e_\parallel})\,\exp(i\omega t)$$

we have

---

[4]Taking divergence of equation (5.37), we have

$$\nabla.(\nabla\times\vec{E}) = 0 = -\nabla.\dot{\vec{B}} = 0 \qquad \frac{\partial}{\partial t}(\nabla.\vec{B}) = 0$$

Thus, $\nabla.\vec{B} = 0$ when it is initially zero. This is equation (5.38). Taking divergence of equation (5.39), we have

$$\nabla.(\nabla\times\vec{B}) = 0 = \mu_0[q_i\nabla.(n_i\,\vec{u_i}) + q_e\nabla.(n_e\,\vec{u_e}) + \frac{\nabla.\dot{\vec{E}}}{c^2}$$

Now, we have

$$\nabla.(n_i\,\vec{u_i}) = -\,\dot{n}_i \qquad \nabla.(n_e\,\vec{u_e}) = -\,\dot{n}_e$$

Using these relations here, we have

$$\mu_0[-q_i\,\dot{n}_i - q_e\,\dot{n}_e + \frac{\nabla.\dot{\vec{E}}}{c^2} = 0 \qquad \frac{\partial}{\partial t}\left[\nabla.\vec{E} - \frac{1}{\epsilon_0}(n_iq_i + n_eq_e)\right] = 0$$

Thus,

$$\nabla.\vec{E} - \frac{1}{\epsilon_0}(n_iq_i + n_eq_e) = 0$$

when it is initially zero. This is equation (5.36).

$$\frac{\partial \vec{u}}{\partial t} = i\omega(u_\perp \hat{e_\perp} + u_\parallel \hat{e_\parallel}) \exp(i\omega t)$$

and

$$\vec{u} \times \vec{B} = u_\perp \exp(i\omega t)\ \hat{e_\perp} \times B\ \hat{e_\parallel} = u_\perp \exp(i\omega t) B\ \hat{e_\perp}$$

Here, we are concerned only with the perpendicular component of velocity and we have

$$\frac{mn(\partial\vec{u}/\partial t)_\perp}{qn\vec{u} \times \vec{B}} = \left|\frac{imn\omega u_\perp \exp(i\omega t)\ \hat{e_\perp}}{qnu_\perp\ \exp(i\omega t) B\ \hat{e_\perp}}\right| = \left|\frac{m\omega}{qB}\right| = \frac{\omega}{\omega_c}$$

For $\omega << \omega_c$, we can neglect the term $mn(\partial \vec{u}/\partial t)$ in comparison to the term $(qn\vec{u} \times \vec{B})$. We shall also neglect the term $(\vec{u}\ .\ \nabla)\ \vec{u}$ and its justification would be discussed later on. For simplification, let us assume that $\vec{E}$ and $\vec{B}$ are uniform, but n and p to have gradient. Then equation (5.40) can be written as

$$0 = qn(\vec{E} + \vec{u}_\perp \times \vec{B}) - \nabla p \tag{5.41}$$

Taking cross product of equation (5.41) with $\vec{B}$, we have

$$0 = qn[\vec{E} \times \vec{B} + (\vec{u}_\perp \times \vec{B}) \times \vec{B}] - \nabla p \times \vec{B}$$

$$0 = qn[\vec{E} \times \vec{B} + \vec{B}\,(\vec{u}_\perp\ .\ \vec{B}) - \vec{u}_\perp\ B^2] - \nabla p \times \vec{B}$$

$$\vec{u}_\perp = \frac{\vec{E} \times \vec{B}}{B^2} - \frac{\nabla p \times \vec{B}}{qnB^2} = \vec{u}_E + \vec{u}_D$$

where

$$\vec{u}_E = \frac{\vec{E} \times \vec{B}}{B^2} \qquad \vec{u}_D = -\frac{\nabla p \times \vec{B}}{qnB^2}$$

The drift $\vec{u_E}$ is the same as discussed in the preceding chapter. There is a new term $\vec{u_D}$ known as the *diamagnetic drift.* Notice that $\vec{u_D}$ is perpendicular to the direction of the gradient in pressure. The neglect of the term $(\vec{u}\ .\nabla)\ \vec{u}$ is justified if $\vec{E} = 0$. If $\vec{E} = -\nabla\phi \neq 0$, the term $(\vec{u}\ .\nabla)\ \vec{u}$ is still zero if $\nabla\phi$ is parallel to ∇p. In case of non-zero value of $(\vec{u}\ .\nabla)\ \vec{u}$, the solution becomes complicated.

With the help of equation (5.34), the diamagnetic drift can be written as

$$\vec{u}_D = -\frac{\gamma p \nabla n \times \vec{B}}{q n^2 B^2}$$

Using $\vec{B} = B\hat{z}$ and $p = nKT$, we have

$$\vec{u}_D = \frac{\gamma KT \hat{z} \times \nabla n}{qnB}$$

For isothermal plasma ($\gamma = 1$) with geometry shown in Figure 5.2, where $\nabla n = n' \, \hat{r}$, we get the following expressions having wide application in Q-machines

$$\vec{u}_{Di} = \frac{KT_i n'}{eBn} \hat{\theta} \qquad \vec{u}_{De} = -\frac{KT_e n'}{eBn} \hat{\theta}$$

The magnetic drift can be easily calculated with the formula

$$u_D = \frac{KT(\mathrm{e\,V})}{B(\mathrm{T})} \frac{1}{\Lambda(\mathrm{m})} \quad \frac{\mathrm{m}}{\mathrm{s}}$$

where $\Lambda = |n/n'|$ is the density scale length.

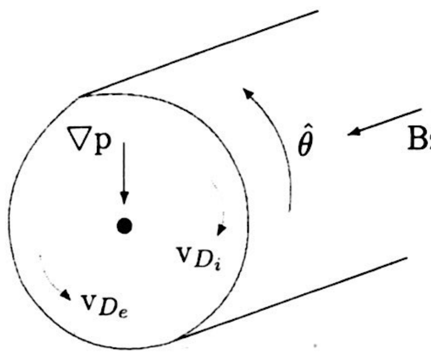

Figure 5.2: Diamagnetic drifts in a cylindrical plasma.

5.4 Fluid-drifts parallel to $\vec{B}$

The component, parallel to $\vec{B}$, of the equation of motion (Equation 5.24) of fluid is

$$mn\left[\frac{\partial u_z}{\partial t} + (\vec{u} \cdot \nabla) u_z\right] = qnE_z - \frac{\partial p}{\partial z} \tag{5.42}$$

Scanned by CamScanner

as the term $\vec{u}_{\parallel} \times \vec{B}$ is zero. For simplicity, let us consider u_z to be spatially uniform, then the term $(\vec{u} \, . \, \nabla)u_z$ is zero and using equation (5.33) in (5.42), we have

$$\frac{\partial u_z}{\partial t} = \frac{q}{m}E_z - \frac{p\gamma}{mn^2}\frac{\partial n}{\partial z}$$

$$= \frac{q}{m}E_z - \frac{\gamma KT}{mn}\frac{\partial n}{\partial z} \tag{5.43}$$

On integrating this equation, we get

$$u_z = \left(\frac{q}{m}E_z - \frac{\gamma KT}{mn}\frac{\partial n}{\partial z}\right)t + u_{z0} \tag{5.44}$$

where u_{z0} is a constant. This shows that the fluid is accelerated along $\vec{B}$ under the combined effect of electrostatic and pressure gradient forces. Equation (5.44) gives the fluid-drift parallel to $\vec{B}$.

Equation (5.43) can·be used for drawing an important result. The electrons, being very light may be considered as massless. For electrons, $m \to 0$, $q = -e$ and $T = T_e$. As the particles are moving along $\vec{B}$, they do not see any effect of it and we can write $\nabla \times \vec{E} = -\dot{\vec{B}} = 0$. Thus, we have $\vec{E} = -\nabla\phi$, where ϕ is the scalar potential. Equation (5.43) can be expressed as

$$m\frac{\partial v_z}{\partial t} = qE_z - \frac{\gamma KT}{n}\frac{\partial n}{\partial z}$$

For electron, we get

$$0 = (-e)\left(-\frac{\partial \phi}{\partial z}\right) - \frac{\gamma KT_e}{n}\frac{\partial n}{\partial z}$$

$$e\,\frac{\partial \phi}{\partial z} = \frac{\gamma\, KT_e}{n}\frac{\partial n}{\partial z} \tag{5.45}$$

Being light in weight, the electrons are so mobile that their heat conduction can be taken almost infinite. We can therefore assume thermal electrons and can take $\gamma = 1$. Now, integration of equation (5.45) gives

$$e\phi = KT_e \ln n + C$$

For the case when $\phi = 0$, $n = n_0$, we have $C = -KT_e \ln n_0$, and thus

$$e\phi = KT_e \ln n - KT_e \ln n_0 = KT_e \ln\left(\frac{n}{n_0}\right)$$

Hence,

$$n = n_0 \exp(e\phi/KT_e) \tag{5.46}$$

It is just the Boltzmann relation for electrons. Equation (5.46) shows that the number of electrons increases exponentially with the increase of the potential ϕ. This expression can be understood that electrons, being light are very mobile and would be accelerated to high energies quickly when they experience a net force.

5.5 Plasma approximation

In usual applications of Maxwell's equations, the electric field $\vec{E}$ is calculated from the Poisson equation

$$\epsilon_0 \nabla . \vec{E} = \sigma$$

for the given charge density σ. A plasma has a tendency to remain neutral; when the positive ions move, the electrons follow them. The $\vec{E}$ must adjust itself to preserve the neutrality. The charge density is of secondary importance in plasmas. Now, $\vec{E}$ is calculated from the equations of motion of the ions and electrons. For the known $\vec{E}$, Poisson equation is used to calculate the charge density. Hence, the use of Poisson equation for plasmas is in a reverse order. This is, however, the case for low-frequency motions in which the electron inertia does not play any role.

In a plasma, it customary to assume $n_i = n_e$ while $\nabla . \vec{E} \neq 0$, though it violates the Poisson equation. This is known as the *plasma approximation.* A fundamental distinction of plasmas is not to use Poisson equation to calculate $\vec{E}$ unless it is unavoidable. In the set of fluid equations, we may remove Poisson equation and also one of the unknowns by setting $n_i = n_e = n$.

The aforesaid plasma approximation is not valid in high-frequency electron waves. Then $\vec{E}$ is found from Maxwell's equations rather than from the equations of motion of ions and electrons.

Regarding the validity of plasma approximation would be discussed when theory of ion waves would be discussed. At that time we shall have a clear picture for the use of Poisson equation in the derivation of Debye shielding.

5.6 Problems and questions

1. Though each gyrating particle in a plasma has its magnetic moment, but the plasma cannot be treated as a magnetic material. Comment on it.

2. Derive equation of motion for an isotropic plasma. How it is modified for partially ionized, anisotropic plasma.

3. Particles in a plasma do not collide so frequently as in an ordinary fluid, but fluid mechanics still works for a plasma. Comment on it.

4. Derive expressions for fluid-drifts perpendicular and parallel to the magnetic field $\vec{B}$

5. Write short notes on the following

   (i) Maxwell's equations

   (ii) Stress tensor for a plasma

   (v) Comparison of ordinary hydrodynamics with plasma dynamics

   (vi) Equation of continuity

   (vii) Equation of state of plasma

   (viii) Complete set of fluid equations

   (ix) Diamagnetic drift

   (x) Fluid-drifts perpendicular to $\vec{B}$

   (x) Fluid-drifts parallel to $\vec{B}$

   (xi) Plasma approximation

6
Waves in a Fluid Plasma

Because of large density, plasma is considered as a fluid. Here, we shall discuss propagation of waves in a fluid plasma

6.1 Representation of waves

A sinusoidally oscillating quantity, say, the particle density n, can be expressed as

$$n = n_0 \, e^{i(\vec{k}.\vec{r} - \omega t)} \tag{6.1}$$

As a convention, the real part of a complex number is a measurable quantity. For example, in equation (6.1), the measurable quantity is

$$n = n_0 \, \cos(\vec{k}.\vec{r} - \omega t) \tag{6.2}$$

This equation (6.2) represents a wave having a constant amplitude n_0, wave-vector $\vec{k}$ and angular frequency ω. When the wave propagation is only in one direction, say, x-direction, the wave-vector $\vec{k}$ has only one component (let us denote it, without any suffix, by k) and thus, the wave equation along the x-direction is

$$n = n_0 \, e^{i(kx - \omega t)} \qquad \text{or} \qquad n = n_0 \cos(kx - \omega t) \tag{6.3}$$

A point of constant phase on the wave moves so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(kx - \omega t\Big) = 0 \qquad k\frac{\mathrm{d}x}{\mathrm{d}t} - \omega = 0 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\omega}{k} \equiv v_\phi$$

Here, v_ϕ is known as the *phase velocity*. When ω/k is positive, the wave propagates in the right direction, that is, the x increases with the increase of t so that $kx - \omega t$ remains constant. When ω/k is negative, the wave propagates in the left direction, that is, the x decreases with

the increase of t so that $kx - \omega t$ remains constant. Equation (6.3) shows that reversing the signs of both ω and k makes no difference.

Let us now consider another sinusoidally oscillating quantity in one direction, say, the electric field $\vec{E}$, expressed as

$$\vec{E}=\vec{E}_0 \; e^{i(kx-\omega t+\delta)} \qquad \text{or} \qquad \vec{E}=\vec{E}_0 \cos(kx - \omega t + \delta)$$

where $\vec{E}_0$ is a real constant vector and δ the phase angle. In equations (6.1) and (6.2), the phase angle was zero. Information about the phase angle can be merged into the amplitude in the following manner

$$\vec{E}=\vec{E}_0 \; e^{i(kx-\omega t+\delta)} = \vec{E}_0 \; e^{i\delta} e^{i(kx-\omega t)} = \vec{E}_c \; e^{i(kx-\omega t)}$$

where $\vec{E}_c$ is a complex amplitude. The phase angle can be obtained from $\vec{E}_c$ as $\mathrm{Re}(\vec{E}_c) = \vec{E}_0 \cos\delta$ and $\mathrm{Im}(\vec{E}_c) = \vec{E}_0 \sin\delta$. Thus,

$$\tan\delta = \frac{\mathrm{Im}(\vec{E}_c)}{\mathrm{Re}(\vec{E}_c)}$$

6.2 Group velocity of a wave

It may be unbelievable, at the first instance, to know that the phase velocity of a wave in a plasma often exceeds the velocity of light c. However, there is no violation of the theory of relativity, because an infinitely long wave of constant amplitude does not carry any information. For example, carrier of a radio wave carries no information until it is modulated. The modulation-information does not travel at the phase velocity, but at the group velocity, which is always smaller than c. Consider a modulated wave formed by addition of the following two waves of nearly equal frequencies.

$$E_1 = E_0 \cos\left[(k + \Delta k)x - (\omega + \Delta\omega)t\right]$$

$$E_2 = E_0 \cos\left[(k - \Delta k)x - (\omega - \Delta\omega)t\right]$$

where E_1 and E_2 differ in frequency by $2\Delta\omega$. Appropriate to the medium, each wave propagating through it must have the phase velocity ω/k. Thus, there should be a difference $2\Delta k$ in the wave-vector. Denoting

$$a = kx - \omega t \qquad \text{and} \qquad b = (\Delta k)x - (\Delta\omega)t \tag{6.4}$$

so that

$$E_1 = E_0 \cos(a+b) \qquad E_2 = E_0 \cos(a-b) \tag{6.5}$$

Addition of these equations gives

$$E_1 + E_2 = E_0 \cos(a+b) + E_0 \cos(a-b) = 2E_0 \cos a \ \cos b \tag{6.6}$$

Using equation (6.4) in (6.6), we have

$$E_1 + E_2 = 2E_0 \cos[(\Delta k)x - (\Delta\omega)t] \cos(kx - \omega t)$$

This is a sinusoidal modulated wave (Figure 6.1) with variable amplitude $2E_0 \cos[(\Delta k)x - (\Delta\omega)t]$, which travels with velocity $\Delta\omega/\Delta k$ and carries information. Taking $\Delta\omega \to 0$ (or $\Delta k \to 0$), group velocity v_g is defined as

$$v_g = \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

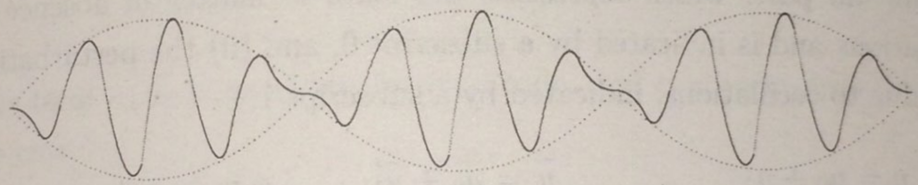

Figure 6 1: Spatial variation of the resultant o f two waves with a frequency difference

6.3 Linearisation of equations

Consider a plasma where a particle has mass m, charge q and average velocity is u in an electric field $\vec{E}$ and magnetic field $\vec{B}$. Let the plasma be homogeneous and isotropic with the particle density be n and kinetic temperature T. In the preceding chapter, we obtained a set of equations. This set of equations can be solved with the help of the process of linearisation of equations. Here, we shall discuss the procedure of linearisation for the following equations:

(a) Equation of motion of the particle

$$mn\left[\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u}\right] = qn(\vec{E} + \vec{u} \times \vec{B}) - \gamma KT\nabla n \tag{6.7}$$

where K is the Boltzmann constant, and γ the ratio of specific heats at constant pressure and constant volume.

(b) Equation of continuity

$$\frac{\partial n}{\partial t} + \nabla.(n\vec{u}) = 0 \tag{6.8}$$

(c) Poisson equation

$$\epsilon_0 \nabla.\vec{E} = e(n_i - n_e) \tag{6.9}$$

where n_i and n_e are densities of positive ions and electrons, respectively, and an ion and an electron carry equal magnitude e of the charge.

The procedure of linearisation is applicable only when amplitude of each oscillating quantity is small, so that the terms having two or higher degrees of amplitude can be neglected. For linearisation, the first step is to separate the variable parameters into two parts: (i) an equilibrium part, which represents the state of matter in absence of oscillations and is indicated by a subscript 0, and (ii) the perturbation part due to oscillations, indicated by a subscript 1:

$$n = n_0 + n_1 \qquad \vec{u} = \vec{u_0} + \vec{u_1} \qquad \phi = \phi_0 + \phi_1$$

$$\vec{E} = \vec{E_0} + \vec{E_1} \qquad \vec{B} = \vec{B_0} + \vec{B_1} \tag{6.10}$$

Before perturbations, the plasma is homogeneous and at rest, so that

$$\nabla n_0 = \nabla\phi_0 = \vec{u_0} = \vec{E_0} = 0 \qquad \frac{\partial n_0}{\partial t} = 0 \tag{6.11}$$

6.3.1 Linearisation of the equation of motion

Using equations (6.10) and (6.11) in (6.7) we get

$$m(n_0 + n_1)\left[\frac{\partial}{\partial t}(\vec{u_0} + \vec{u_1}) + \left\{(\vec{u_0} + \vec{u_1}).\nabla\right\}(\vec{u_0} + \vec{u_1})\right]$$

$$= q(n_0 + n_1)\left[(\vec{E_0} + \vec{E_1}) + (\vec{u_0} + \vec{u_1}) \times (\vec{B_0} + \vec{B_1})\right] - \gamma KT\nabla(n_0 + n_1)$$

so that

$$m(n_0 + n_1)\Big[\frac{\partial \vec{u_1}}{\partial t} + (\ \vec{u_1}\ .\nabla)\ \vec{u_1}\ \Big] = q(n_0 + n_1)\Big[\ \vec{E_1} + \vec{u_1}\ \times(\vec{B_0} + \vec{B_1})\Big]$$

$$-\gamma KT\nabla n_1 \tag{6.12}$$

Neglecting the quadratic and higher degree terms in equation (6.12), we have

$$mn_0\frac{\partial \vec{u_1}}{\partial t} = qn_0(\vec{E_1} + \vec{u_1}\ \times\ \vec{B_0}) - \gamma KT\nabla n_1$$

6.3.2 Linearisation of the equation of continuity

Using equations (6.10) and (6.11) in (6.8) we get

$$\frac{\partial}{\partial t}(n_0 + n_1) + \nabla.[(n_0 + n_1)(\vec{u_0} + \vec{u_1})] = 0$$

$$\frac{\partial n_1}{\partial t} + \nabla.[n_0\ \vec{u_1}\ + n_1\ \vec{u_1}] = 0$$

The term $n_1\vec{u_1}$ is quadratic in amplitude, and can be neglected. Now, we have

$$\frac{\partial n_1}{\partial t} + \nabla.(n_0\ \vec{u_1}) = 0 \qquad\qquad \frac{\partial n_1}{\partial t} + n_0\nabla.\ \vec{u_1} = 0$$

6.3.3 Linearisation of Poisson equation

Using equations (6.10) and (6.11) in (6.9) we get

$$\epsilon_0\nabla.(\vec{E_0} + \vec{E_1}) = e[n_{i0} - (n_{e0} + n_{e1})]$$

$$\epsilon_0\nabla.\ \vec{E_1} = -e\ n_{e1}$$

where we have used $n_i = n_{i0}$ as ions are stationary, $n_e = n_{e0} + n_{e1}$, and $n_{i0} = n_{e0}$ (as in the equilibrium, the neutrality is maintained).

6.4 Plasma oscillations

In an equilibrium position, charged particles in a plasma are uniformly distributed in such a manner that neutrality is maintained everywhere. When electrons in the plasma are displaced relative to the uniform background of the ions, an electric field is developed in such a direction that it tries to pull the electrons back to restore the neutrality. While returning back, owing to inertia, the electrons overshoot and now the electric field is developed in the reverse direction which tries to pull back the electrons to their positions of equilibrium. Thus, the electrons oscillate about their equilibrium positions with a characteristic frequency, known as the *plasma frequency*. Because of low mass of electrons, these oscillations are so fast that the massive ions are not capable to respond the oscillating field, generated by the oscillations of electrons. Hence, the ions may be treated as fixed.

An expression for the plasma frequency ω_p can be derived in a simple manner with the help of the following assumptions:

(i) The plasma has a quite large extent.

(ii) There is no magnetic field, so that from the Maxwell equation, we have

$$\nabla\times \vec{E} = -\dot{\vec{B}} = 0 \qquad \text{and hence,} \qquad \vec{E} = -\nabla\phi$$

where ϕ is a scalar quantity.

(iii) The ions are fixed and form a uniform background in the space.

(iv) There are no thermal motions.

(v) Due to electric field generated in the plasma, motion of the electrons are along one direction, say, the x direction. Thus, we have

$$\nabla \equiv \frac{\partial}{\partial x}\,\hat{i} \qquad \text{and} \qquad \vec{E} = E\,\hat{i}$$

The equations of motion and continuity for an electron are

$$m_e n_e\left[\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e\,.\nabla)\vec{u}_e\right] = -en_e\vec{E} \tag{6.13}$$

$$\frac{\partial n_e}{\partial t} + \nabla.(n_e\vec{u_e}) = 0 \tag{6.14}$$

Here, we are assuming collision-less, homogeneous and isotropic plasma. These are high frequency oscillations, generated due to deviation from neutrality and therefore, we can use Poisson equation for one-dimensional motion

$$\epsilon_0\frac{\partial E}{\partial x} = e(n_i - n_e) \tag{6.15}$$

where n_i and n_e are densities of ions and electrons, respectively. When amplitude of each oscillating quantity is small, the set of equations (6.13) – (6.15) can be solved by using the procedure of the linearisation, where the terms having two and higher powers of amplitude are neglected. For this, let us first separate the variable parameters into two parts: (i) an equilibrium part, which represents the state of plasma in absence of oscillations, and is indicated by a subscript 0 and (ii) the perturbation part due to oscillations, indicated by a subscript 1:

$$n_e = n_0 + n_1 \qquad \vec{u}_e = \vec{u_0} + \vec{u_1} \qquad \vec{E} = \vec{E_0} + \vec{E_1} \tag{6.16}$$

Before displacement of electrons, plasma is homogeneous and at rest, so that

$$\nabla n_0 = \vec{u_0} = \vec{E_0} = 0 \qquad \frac{\partial n_0}{\partial t} = \frac{\partial \vec{u_0}}{\partial t} = 0 \tag{6.17}$$

Using equations (6.16) and (6.17) in (6.13) - (6.15) and on linearisation, we get

$$m_e\frac{\partial \vec{u_1}}{\partial t} = -e\vec{E_1}; \qquad \frac{\partial n_1}{\partial t} + n_0\nabla.\,\vec{u_1} = 0; \qquad \epsilon_0\frac{\partial E_1}{\partial x} = -en_1 \tag{6.18}$$

For convenience, we consider one-dimensional case and assume that the oscillating quantities behave sinusoidally as $e^{i(kx-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by ik. From the equations (6.18), we get

$$-im_e\omega u_1 = -eE_1 \tag{6.19}$$

$$-i\omega n_1 = -n_0 iku_1 \tag{6.20}$$

$$ik\epsilon_0 E_1 = -en_1 \tag{6.21}$$

Equations (6.19) – (6.21) can be rearranged as

$$-im_e\omega u_1 = -e\left(\frac{-e}{ik\epsilon_0}\right)n_1 = \left(\frac{e^2}{ik\epsilon_0}\right)\left(\frac{n_0 k}{\omega}\right)u_1 = -i\left(\frac{n_0 e^2}{\epsilon_0\omega}\right)u_1$$

so that

$$\left(m_e\omega - \frac{n_0 e^2}{\epsilon_0\omega}\right)u_1 = 0$$

Since u_1 is not zero, we have

$$\omega^2 = \frac{n_0 e^2}{m_e\epsilon_0}$$

Thus, the plasma frequency ω_p is

$$\omega_p = \left(\frac{n_0 e^2}{m_e\epsilon_0}\right)^{1/2} \text{ rad/sec} \tag{6.22}$$

Plasma frequency depending on the plasma density is also a fundamental parameter for a plasma. Since mass of electron is very low, plasma frequency is very high. Using the values of various parameters, plasma frequency (Hz) for electron is given by

$$f_p = \omega_p/2\pi \approx 9\sqrt{n}$$

where n is the electron density per m^3. For a plasma having electron density $n = 10^{18}$ m^{-3}, we have

$$f_p = 9\left(10^{18}\right)^{1/2} = 9 \times 10^9 \text{ Hz} = 9 \text{ GHz}$$

Hence, the plasma frequency generally lies in the microwave region.[1] Equation (6.22) shows that ω does not depend on k, and therefore, the group velocity $d\omega/dk$ is zero. Thus, no information transmits through these waves from one region to another.

Electromagnetic radiation of frequency smaller than the plasma frequency ($\omega < \omega_p$), on incidence on the plasma are reflected back, whereas the radiation of frequency larger than the plasma frequency ($\omega > \omega_p$) transmit through the plasma. This property of plasma in the ionosphere around the earth has been exploited for communication purposes. Obviously, in order to communicate with the geostationary satellites the frequency of the signal must be larger than the plasma frequency of the ionosphere.

[1]Cyclotron frequency for electron is $f_{ce} \approx 28$ GHz/Tesla. Hence, when $B \approx 0.32$ Tesla and $n = 10^{18}$ m^{-3}, plasma and cyclotron frequencies for electrons are equal.

6.5 Electron plasma waves

In the previous section, we assumed no thermal motions and found that the electrons oscillate about their equilibrium positions with frequency ω_p which is independent of k. Thus, the group velocity is zero, and hence, there is no transmission of information from one place to another. When the thermal motions are included, it provides propagation of plasma oscillations. Electrons moving with their thermal velocities into adjacent layers of plasma can carry information regarding the happening in the previous oscillating region. Propagation of such information is known as the *electron plasma waves.* Such waves can be obtained through inclusion of the term $-\nabla p_e$, due to thermal motions of electrons, in the equation of motion.

If we take all the assumptions of the previous section except the assumption (iv), (*i.e.*, here we include thermal motions), the equations of motion and continuity for an electron are

$$m_e n_e\left[\frac{\partial \vec{u_e}}{\partial t} + (\vec{u_e}.\nabla)\vec{u_e}\right] = -en_e\vec{E} - \nabla p_e \tag{6.23}$$

$$\frac{\partial n_e}{\partial t} + \nabla.(n_e\vec{u_e}) = 0 \tag{6.24}$$

Here, we are assuming collision-less, homogeneous and isotropic plasma. These are high frequency oscillations which are generated due to deviation from neutrality and therefore, we can use Poisson equation for one-dimensional motion

$$\epsilon_0\frac{\partial E}{\partial x} = e(n_i - n_e) \tag{6.25}$$

where n_i and n_e are densities of ions and electrons, respectively. When amplitude of each oscillating quantity is small, the set of equations (6.23) – (6.25) can be solved by using the procedure of linearisation, where the terms having two and higher powers of amplitude are neglected. For this, let us first separate the variable parameters into two parts (one-dimensional case): (i) an equilibrium part, which represents the state of plasma in absence of oscillations, and is indicated by a subscript 0 and (ii) the perturbation part due to oscillations, indicated by a subscript 1:

$$n_e = n_0 + n_1 \qquad u_e = u_0 + u_1 \qquad E = E_0 + E_1 \tag{6.26}$$

Before displacement of electrons, plasma is homogeneous and at rest, so that

$$\frac{\partial n_0}{\partial x} = u_0 = E_0 = 0 \qquad \frac{\partial n_0}{\partial t} = 0 \tag{6.27}$$

The term ∇p_e for one-dimensional space can be written as

$$\frac{\partial p_e}{\partial x} = \frac{\gamma p_e}{n_e}\frac{\partial n_e}{\partial x} = \frac{\gamma n_e K T_e}{n_e}\frac{\partial n_e}{\partial x} = \gamma K T_e \frac{\partial}{\partial x}(n_0 + n_1) = \gamma K T_e \frac{\partial n_1}{\partial x} \tag{6.28}$$

For one-dimensional space we have

$$\gamma = 1 + \frac{2}{N} = 1 + \frac{2}{1} = 3$$

Using equations (6.26) – (6.28) in (6.23) – (6.25) and on linearisation, we get

$$m_e n_0 \frac{\partial u_1}{\partial t} = -e n_0 E_1 - 3KT_e \frac{\partial n_1}{\partial x} \tag{6.29}$$

$$\frac{\partial n_1}{\partial t} + n_0 \frac{\partial u_1}{\partial x} = 0 \tag{6.30}$$

$$\epsilon_0 \frac{\partial E_1}{\partial x} = -e n_1 \tag{6.31}$$

For convenience, we consider one-dimensional case and assume that the oscillating quantities behave sinusoidally as $e^{i(kx-\omega t)}$.. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient $\partial/\partial x$ by ik. From the equations (6.29) – (6.31), we get

$$-i m_e n_0 \omega u_1 = -e n_0 E_1 - 3KT_e i k n_1 \tag{6.32}$$

$$-i\omega n_1 = -n_0 i k u_1 \tag{6.33}$$

$$ik\epsilon_0 E_1 = -e n_1 \tag{6.34}$$

Equations (6.32) – (6.34) can be rearranged as

$$-i m_e n_0 \omega u_1 = -e n_0 \left(\frac{-e}{ik\epsilon_0}\right) n_1 - 3KT_e i k n_1$$

$$= -i\left(\frac{n_0 e^2}{k\epsilon_0} + 3KT_e k\right) n_1 = -i\left(\frac{n_0 e^2}{k\epsilon_0} + 3KT_e k\right)\frac{n_0 k}{\omega} u_1$$

so that

$$\left[\omega^2 - \frac{n_0 e^2}{m_e \epsilon_0} - \frac{3KT_e k^2}{m_e}\right] u_1 = 0$$

Since u_1 is not zero, we have

$$\omega^2 = \frac{n_0 e^2}{m_e \epsilon_0} + \frac{3KT_e k^2}{m_e}$$

$$= \omega_p^2 + \frac{3}{2}\, k^2 u_{th}^2$$

where $\omega_p (\equiv n_0 e^2 / m_e \epsilon_0)$ is the plasma frequency discussed earlier and $u_{th} (= \sqrt{2KT_e/m_e})$ is the thermal velocity in one-dimensional space. Figure 6.2 shows a plot of the dispersion relation–a variation of ω versus k.

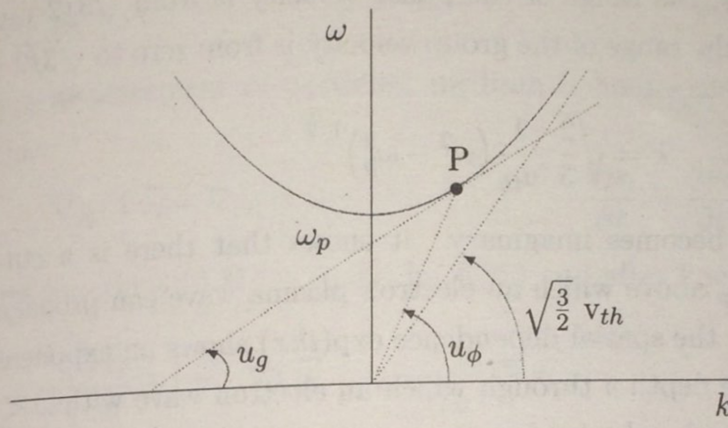

Figure 6.2: Variation of ω versus k for electron plasma waves

At a point P on the curve, the slope of the line joining the point and the origin gives the phase velocity u_ϕ, whereas the tangent to the curve at P gives the group velocity u_g. We can write

$$\frac{\omega^2}{k^2} = \frac{\omega_p^2}{k^2} + \frac{3}{2}\, u_{th}^2$$

$$\frac{\omega}{k} = \sqrt{\frac{3}{2}}\, u_{th} \left(1 + \frac{2\,\omega_p^2}{3k^2 u_{th}^2}\right)^{1/2} \equiv u_\phi$$

It shows that the phase velocity u_ϕ is never smaller than $\sqrt{3/2}\ u_{th}$. When $k \to \infty$, the phase velocity $u_\phi \to \sqrt{3/2}\ u_{th}$. The group velocity u_g is obtained in the following manner

$$2\omega\ \mathrm{d}\omega = \frac{3}{2}\, u_{th}^2 2k\ \mathrm{d}k$$

Hence, the group velocity is

$$u_g = \frac{d\omega}{dk} = \frac{3}{2}\frac{k}{\omega}u_{th}^2 = \frac{3}{2}\frac{u_{th}^2}{u_\phi}$$

$$= \sqrt{\frac{3}{2}}\, u_{th}\Big(1 + \frac{2\omega_p^2}{3k^2u_{th}^2}\Big)^{-1/2}$$

It shows that the group velocity u_g is never larger than $\sqrt{3/2}\,u_{th}$. When $k \to \infty$, the group velocity $u_g \to \sqrt{3/2}\,u_{th}$.

When we move along the dispersion curve from a large value of k towards the point $k = 0$, the phase velocity varies from $\sqrt{3/2}\,u_{th}$ to the infinite value whereas the group velocity varies from $\sqrt{3/2}\,u_{th}$ to the zero value. Thus, the range of the phase velocity is from $\sqrt{3/2}\,u_{th}$ to infinite whereas the range of the group velocity is from zero to $\sqrt{3/2}\,u_{th}$. We can write

$$k = \sqrt{\frac{2}{3}}\,\frac{1}{u_{th}}\Big(\omega^2 - \omega_p^2\Big)^{1/2}$$

For $\omega < \omega_p$, k becomes imaginary. it shows that there is a cut-off frequency $\omega = \omega_p$ above which an electron plasma wave can propagate. For imaginary k, the spatial dependence $\exp(ikx)$ shows an exponential attenuation. The depth δ through which an electron wave with $\omega < \omega_p$ can propagate can be obtained as

$$e^{ikx} = e^{-|k|x} = e^{-k/\delta} \qquad\qquad \delta = |k|^{-1} = \sqrt{\frac{3}{2}}\,\frac{u_{th}}{(\omega_p^2 - \omega^2)^{1/2}}$$

6.6 Sound waves

After discussion of electron waves, we want to discuss about ion waves. Before discussion of ion waves, however, let us first discuss about the sound waves, which take place in an ordinary medium. Neglecting the viscosity, the equation of motion is

$$\rho\Big[\frac{\partial \vec{u}}{\partial t} + (\,\vec{u}\,.\nabla)\,\vec{u}\,\Big] = -\nabla p \tag{6.35}$$

and the equations of continuity and state are

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\,\vec{u}) = 0 \qquad \text{and} \qquad p = C\rho^\gamma \tag{6.36}$$

where ρ is the mass density. We have

$$\nabla p = \frac{\gamma p}{\rho} \nabla \rho \tag{6.37}$$

When amplitude of each oscillating quantity is small, the set of equations (6.35) – (6.36) can be solved by using the procedure of linearisation, where the terms having two and higher powers of amplitude are neglected. For this, let us first separate the variable parameters into two parts: (i) an equilibrium part, which represents the state of matter in absence of oscillations, and is indicated by a subscript 0 and (ii) the perturbation part due to oscillations, indicated by a subscript 1:

$$\rho = \rho_0 + \rho_1 \qquad \vec{u} = \vec{u_0} + \vec{u_1} \qquad p = p_0 + p_1 \tag{6.38}$$

Before displacement of particles, medium is homogeneous and at rest, so that

$$\nabla \rho_0 = \vec{u_0} = \nabla p_0 = 0 \qquad \frac{\partial \rho_0}{\partial t} = \frac{\partial \vec{u_0}}{\partial t} = 0 \tag{6.39}$$

Using equations (6.37) – (6.39) in (6.35) and after linearisation, we get

$$\rho_0 \left[\frac{\partial \vec{u_1}}{\partial t} + (\vec{u_1} \, .\nabla) \, \vec{u_1} \right] = -\frac{\gamma \, p_0}{\rho_0} \, \nabla \rho_1$$

The term $(\vec{u_1} \, .\nabla) \, \vec{u_1}$ is quadratic in amplitude, and can be neglected. Now, we have

$$\rho_0 \frac{\partial \vec{u_1}}{\partial t} = -\frac{\gamma p_0}{\rho_0} \nabla \rho_1 \tag{6.40}$$

Using equations (6.38) – (6.39) in first of (6.36) and after linearisation, we get

$$\frac{\partial \rho_1}{\partial t} + \nabla . \, (\rho_0 \, \vec{u_1}) = 0 \qquad \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla . \, \vec{u_1} = 0 \tag{6.41}$$

For convenience, we consider one-dimensional case and assume that the oscillating quantities behave sinusoidally as $e^{i(kx - \omega t)}$. Then the time derivative $(\partial / \partial t)$ can be replaced by $-i\omega$, and the gradient $(\partial / \partial x)$ by ik. Now, equations (6.40) and (6.41) become

$$-i\omega \rho_0 u_1 = -\frac{\gamma p_0}{\rho_0} ik \rho_1 \tag{6.42}$$

$$-i\omega \rho_1 + \rho_0 ik u_1 = 0 \tag{6.43}$$

Equations (6.42) and (6.43) can be rearranged as

$$\left[\omega^2 - \frac{\gamma\, p_0}{\rho_0} k^2\right] u_1 = 0$$

Since u_1 is not zero, we have

$$\omega^2 = \frac{\gamma\, p_0}{\rho_0} k^2 \qquad\qquad \frac{\omega}{k} = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}$$

Using $p_0 = n_0 KT$ and $\rho_0 = n_0 m_p$, where m_p is mass of a proton, we get

$$\frac{\omega}{k} = \left(\frac{\gamma KT}{m_p}\right)^{1/2} \equiv c_s$$

where c_s is the velocity of sound wave in a neutral gas. The waves are propagating from one layer to another through collisions among the gas molecules, and hence they are pressure waves. For one-dimensional case, $\gamma = 3$, and thus, the phase velocity v_ϕ and group velocity v_g of the wave are

$$v_\phi = \frac{\omega}{k} = \sqrt{\frac{3}{2}}\, u_{th} \qquad \text{and} \qquad v_g = \frac{d\omega}{dk} = \sqrt{\frac{3}{2}}\, u_{th}$$

where $u_{th} = \sqrt{2KT/m_p}$ is the average thermal velocity of atoms in the gas. It is interesting to find that in a fully ionized plasma (*i.e.*, in absence of collisions), analogous waves, called the 'ion acoustic waves', or simply, the 'ion waves' are found.

6.7 Ion waves

In a fully ionized plasma, in absence of collisions, sound waves do not occur. Ions, being charged particles, can transmit vibrations to each other. Since ions are massive particles, these vibrations will be low frequency oscillations. In absence of magnetic field $\vec{B}$, from the Maxwell equation, we have

$$\nabla\times \vec{E} = -\dot{\vec{B}} = 0 \qquad \text{and hence,} \qquad \vec{E} = -\nabla\phi$$

where ϕ is a scalar quantity. The equations of motion and continuity for an ion are

$$m_i n_i\Big[\frac{\partial \vec{u_i}}{\partial t} + (\vec{u_i}.\nabla)\vec{u_i}\Big] = en_i\vec{E} - \nabla p_i \tag{6.44}$$

$$\frac{\partial n_i}{\partial t} + \nabla.(n_i\vec{u_i}) = 0 \tag{6.45}$$

From the equation of state $p_i = C\rho_i^{\gamma_i}$, we have

$$\nabla p_i = \gamma_i p_i \frac{\nabla n_i}{n_i} = \frac{\gamma_i n_i K T_i}{n_i}\nabla n_i = \gamma_i K T_i \nabla n_i$$

Here, we have used $p_i = n_i K T_i$. Hence, equation (6.44) of motion can be written as

$$m_i n\Big[\frac{\partial \vec{u_i}}{\partial t} + (\vec{u_i}.\nabla)\vec{u_i}\Big] = -en\nabla\phi - \gamma_i K T_i \nabla n \tag{6.46}$$

where we have used the plasma approximation $n_i = n_e = n$, and therefore, we shall not use the Poisson equation. When amplitude of each oscillating quantity is small, let us linearize the equation (6.46) by neglecting the terms having two and higher powers of amplitude. For this, let us first separate the variable parameters into two parts: (i) an equilibrium part, which represents the state of plasma in absence of oscillations, and is indicated by a subscript 0 and (ii) the perturbation part due to oscillations, indicated by a subscript 1:

$$n = n_0 + n_1 \qquad \vec{u_i} = \vec{u_0} + \vec{u_1} \qquad \phi = \phi_0 + \phi_1 \tag{6.47}$$

Before displacement of electrons, plasma is homogeneous and at rest, so that

$$\nabla n_0 = \vec{u_0} = \nabla\phi_0 = 0 \qquad \frac{\partial n_0}{\partial t} = 0 \tag{6.48}$$

Using equations (6.47) and (6.48) in (6.46) and on linearisation, we get

$$m_i n_0 \frac{\partial \vec{u_1}}{\partial t} = -en_0\nabla\phi_1 - \gamma_i K T_i \nabla n_1 \tag{6.49}$$

Fluid drift parallel to $\vec{B}$ is the same as in absence of $\vec{B}$. For the electric field $\vec{E} = -\nabla\phi = -\nabla(\phi_0+\phi_1) = -\nabla\phi_1$, the density of massless electrons is (mass of an electron may be considered as negligible in comparison to that of proton)

$$n_e = n = n_0 \exp\Big(\frac{e\phi_1}{KT_e}\Big)$$

$$= n_0\Big(1 + \frac{e\phi_1}{KT_e} + \dots\Big) = n_0\Big(1 + \frac{e\phi_1}{KT_e}\Big)$$

Neglecting higher order terms, we have

$$n - n_0 = n_0 \frac{e\phi_1}{KT_e} \qquad\qquad n_1 = n_0 \frac{e\phi_1}{KT_e} \tag{6.50}$$

Equation (6.49) can be written as

$$m_i n_0 \frac{\partial \vec{u_1}}{\partial t} = -en_0 \nabla\phi_1 - \frac{\gamma_i n_0 e T_i}{T_e} \nabla\phi_1 \tag{6.51}$$

Using equations (6.47) – (6.48) in (6.45) and on linearisation, we get

$$\frac{\partial n_1}{\partial t} + n_0 \nabla . \, \vec{u_1} = 0 \tag{6.52}$$

For convenience, we consider one-dimensional case and assume that the oscillating quantities behave sinusoidally as $e^{i(kx-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by ik. From the equations (6.50) – (6.52), we get

$$n_1 = n_0 \frac{e\phi_1}{KT_e} \tag{6.53}$$

$$-im_i n_0 \omega u_1 = -en_0 ik\phi_1 - \frac{\gamma_i n_0 e T_i}{T_e} ik\phi_1 \tag{6.54}$$

$$-i\omega n_1 + n_0 iku_1 = 0 \tag{6.55}$$

Equations (6.53) – (6.55) can be rearranged as

$$-im_i n_0 \omega u_1 = -iken_0 \Big(1 + \frac{\gamma_i T_i}{T_e}\Big)\phi_1$$

$$= -iken_0 \Big(1 + \frac{\gamma_i T_i}{T_e}\Big)\frac{KT_e}{n_0 e} n_1 = -ikK(T_e + \gamma_i T_i)\frac{n_0 k}{\omega} u_1$$

so that

$$\Big[\omega^2 - \frac{k^2 K}{m_i}\Big(T_e + \gamma_i T_i\Big)\Big] u_1 = 0$$

Since u_1 is not equal to zero, we have

$$\omega^2 = \frac{k^2 K}{m_i}\Big(T_e + \gamma_i T_i\Big)$$

$$\frac{\omega}{k} = \Big(\frac{K\,T_e + \gamma_i K T_i}{m_i}\Big)^{1/2} \equiv v_s \tag{6.56}$$

This is the dispersion relation for ion acoustic waves. Here, v_s is the sound speed in a plasma. Obviously, v_s is equal to the phase velocity as well as the group velocity of the wave. When the ion temperature T_i tends to zero, ion wave still exist and its velocity is

$$v_s = \left(\frac{KT_e}{m_i}\right)^{1/2}$$

This does happen in a neutral gas.

6.8 Validity of plasma approximation ✓

In the preceding section, in the derivation of the velocity of ion waves, we have used the neutrality condition $n_i = n_e$, and have taken finite value for the electric field $\vec{E}$. It is in violation of the Poisson equation. For estimating the error introduced in the process, let us assume $n_i \neq n_e$ and consider the Poisson equation for the hydrogen plasma

$$\epsilon_0 \nabla . \vec{E} = e(n_i - n_e)$$

In absence of $\vec{B}$, we have $\vec{E} = -\nabla\phi$, and the Poisson equation is

$$-\epsilon_0 \nabla^2 \phi = e(n_i - n_e) \tag{6.57}$$

When amplitude of each oscillating quantity is small, let us linearize the equation (6.57) by neglecting the terms having two and higher powers of amplitude. For this, let us first separate the variable parameters into two parts: (i) an equilibrium part, which represents the state of plasma in absence of oscillations, and is indicated by a subscript 0 and (ii) the perturbation part due to oscillations, indicated by a subscript 1:

$$n_i = n_{i0} + n_{i1} \qquad n_e = n_{e0} + n_{e1} \qquad \phi = \phi_0 + \phi_1 \tag{6.58}$$

Using equation (6.58) in (6.57) we have

$$-\epsilon_0 \nabla^2 (\phi_0 + \phi_1) = e\ (n_{i0} + n_{i1} - n_{e0} - n_{e1}) \tag{6.59}$$

Before displacement of electrons, plasma is homogeneous and at rest, so that

$$-\epsilon_0 \nabla^2 \phi_0 = e(n_{i0} - n_{e0}) \tag{6.60}$$

Subtracting equation (6.60) from (6.59), we get

$$-\epsilon_0 \nabla^2 \phi_1 = e(n_{i1} - n_{e1}) \tag{6.61}$$

Fluid drift parallel to $\vec{B}$ is the same as in absence of $\vec{B}$. The density of massless electrons is (mass of an electron may be considered as negligible in comparison to that of proton)

$$n_e = n_0 \exp\left(\frac{e\phi}{KT_e}\right)$$

$$n_{e0} + n_{e1} = n_0 \exp\left(\frac{e(\phi_0 + \phi_1)}{KT_e}\right) \tag{6.62}$$

We have

$$n_{e0} = n_0 \exp\left(\frac{e\phi_0}{KT_e}\right) \tag{6.63}$$

Subtracting equation (6.63) from (6.62), we get

$$n_{e1} = n_0 \exp\left(\frac{e(\phi_0 + \phi_1)}{KT_e}\right) - n_0 \exp\left(\frac{e\phi_0}{KT_e}\right)$$

$$= n_0 \exp\left(\frac{e\phi_0}{KT_e}\right)\left[1 + \frac{e\phi_1}{KT_e} + \ldots - 1\right] = n_{e0}\,\frac{e\phi_1}{KT_e}$$

For convenience, we consider one-dimensional case and assume that the oscillating quantities behave sinusoidally as $e^{i(kx-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient $\partial/\partial x$ by ik. From equation (6.61), we get

$$\epsilon_0 k^2 \phi_1 + e n_{e0}\frac{e\phi_1}{KT_e} = e n_1 \qquad \epsilon_0 \phi_1 \left(k^2 + \frac{n_{e0}e^2}{\epsilon_0 K T_e}\right) = e n_1$$

$$\epsilon_0 \phi_1 \left(k^2 + \frac{1}{\lambda_D^2}\right) = e n_1 \qquad \epsilon_0 \phi_1 \left(k^2 \lambda_D^2 + 1\right) = e n_1 \lambda_D^2 \tag{6.64}$$

Equations (6.54), (6.55) and (6.64) can be rearranged as

$$-i m_i n_0 \omega u_1 = -ikerno\left[1 + \frac{\gamma_i T_i}{T_e}\right]\phi_1 = -iken_0\left[1 + \frac{\gamma_i T_i}{T_e}\right]\left[\frac{e\lambda_D^2}{\epsilon_0(1 + k^2\lambda_D^2)}\right]n_1$$

$$= -iken_0\left[1 + \frac{\gamma_i T_i}{T_e}\right]\left[\frac{e}{\epsilon_0(1 + k^2\lambda_D^2)}\right]\left[\frac{\epsilon_0 K T_e}{n_{e0}e^2}\right]\left[\frac{n_0 k}{\omega}\right]u_1$$

so that

$$\left[\omega^2 - \frac{k^2K}{m_i}\left(T_e + \gamma_i T_i\right)\frac{1}{(1+k^2\lambda_D^2)}\right]u_1 = 0$$

Since u_1 is not equal to zero, we have

$$\omega^2 = \frac{k^2K}{m_i}\left(T_e + \gamma_i T_i\right)\frac{1}{(1+k^2\lambda_D^2)}$$

$$\frac{\omega}{k} = \left(\frac{KT_e + \gamma_i KT_i}{m_i}\right)^{1/2}\frac{1}{\sqrt{(1+k^2\lambda_D^2)}} \tag{6.65}$$

Comparison of equations (6.65) and (6.56) shows that in our plasma approximation ($n_i = n_e$), an error of the order of $k^2\lambda_D^2$ ($= \{2\pi\lambda_D/\lambda\}^2$) is introduced. Since λ_D in comparison to λ is very small in most of the cases, the plasma approximation is valid, except for the cases of the shortest wavelengths.

6.9 Some definitions for waves

In the discussion so far, we did not account for the magnetic field $\vec{B}$. In the presence of $\vec{B}$, several other types of waves are possible. Let us first introduce some terminology for the waves.

(i) Parallel and perpendicular is used to denote the direction of $\vec{k}$ relative to the unperturbed magnetic field $\vec{B_0}$.

(ii) Longitudinal and transverse is used to denote the direction of $\vec{k}$ relative to the perturbed (oscillating) electric field $\vec{E_1}$.

(iii) The wave is termed electrostatic when the oscillating magnetic field $\vec{B_1}$ is zero.

(iv) The wave is termed electromagnetic when the oscillating magnetic field $\vec{B_1}$ is finite (non-zero).

Let us consider the Maxwell equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6.66}$$

and resolve $\vec{E}$ and $\vec{B}$ into two parts: (i) an equilibrium part, which represents the state of plasma in absence of oscillations, and indicated by a subscript 0 and (ii) the perturbation part due to oscillations, indicated by a subscript 1:

$$\vec{E} = \vec{E_0} + \vec{E_1} \qquad\qquad \vec{B} = \vec{B_0} + \vec{B_1} \tag{6.67}$$

Using equation (6.67) in (6.66), we have

$$\nabla \times (\vec{E_0} + \vec{E_1}) = -\frac{\partial \vec{B_0}}{\partial t} - \frac{\partial \vec{B_1}}{\partial t}$$

Since the equilibrium parts do not depend on time as well as on the space coordinates, we have

$$\nabla \times \vec{E_1} = -\frac{\partial \vec{B_1}}{\partial t} \tag{6.68}$$

Let us assume that the oscillating quantities behave sinusoidally as

$$\vec{E_1} = \vec{E_m}\ e^{i(\vec{k}.\vec{r} - \omega t)} \qquad\qquad \vec{B_1} = \vec{B_m}\ e^{i(\vec{k}.\vec{r} - \omega t)} \tag{6.69}$$

Using equation (6.69) in (6.68), we have

$$i\ \vec{k} \times \vec{E_1} = -(-i\omega)\ \vec{B_1} \qquad\qquad \vec{k} \times \vec{E_1} = \omega\ \vec{B_1}$$

When a wave is longitudinal, *i.e.*, $\vec{k}$ is parallel to $\vec{E_1}$, we have $\vec{k} \times \vec{E_1} = 0$, and $\vec{B_1} = 0$ showing that the wave is electrostatic also. When a wave is transverse, *i.e.*, $\vec{k}$ is perpendicular to $\vec{E_1}$, we have $\vec{k} \times \vec{E_1} \neq 0$, and $\vec{B_1} \neq 0$ showing that the wave is electromagnetic also.

In general, $\vec{k}$ may have an arbitrary angle to $\vec{B_0}$ and $\vec{E_1}$, then the wave would have a combination of the principal modes discussed above in (i) – (iv).

6.10 Electrostatic electron waves perpendicular to $\vec{B}$

In an equilibrium position, charged particles in a plasma are uniformly distributed in such a manner that neutrality is maintained. When electrons in a plasma are displaced relative to the uniform background of

ions, an electric field is developed in such a direction that it tries to pull the electrons back to restore neutrality of the plasma. While returning back, owing to inertia, the electrons overshoot and now the electric field is developed in the reverse direction which tries to pull back the electrons to their positions of equilibrium. Thus, the electrons oscillate about their position of equilibrium. Further, owing to the presence of magnetic field, the motions of electrons are not linear. Because of low mass of electrons, their oscillations are so fast that the massive ions are not capable to respond the oscillating field, generated by the oscillations of the electrons. Hence, the ions may be treated as fixed forming a uniform background. We assume also that there are no thermal motions $(T_e = 0)$. The equations of motion and continuity for an electron are

$$m_e n_e \left[\frac{\partial \vec{u_e}}{\partial t} + (\vec{u_e} . \nabla) \vec{u}_e \right] = -e n_e (\vec{E} + \vec{u}_e \times \vec{B}) \tag{6.70}$$

$$\frac{\partial n_e}{\partial t} + \nabla . (n_e \vec{u_e}) = 0 \tag{6.71}$$

Here, we are assuming collision-less, homogeneous and isotropic plasma. These are high frequency oscillations which are generated due to deviation from neutrality and therefore, we can use Poisson equation

$$\epsilon_0 \nabla . \vec{E} = e(n_i - n_e) \tag{6.72}$$

where n_i and n_e are densities of ions and electrons, respectively. When amplitude of each oscillating quantity is small, the set of equations (6.70) – (6.72) can be solved by using the procedure of linearisation, where the terms having two and higher powers of amplitude are neglected. For this, let us first separate the variable parameters into two parts: (i) an equilibrium part, which represents the state of plasma in absence of oscillations, and is indicated by a subscript 0 and (ii) the perturbation part due to oscillations, indicated by a subscript 1:

$$n_e = n_0 + n_1 \qquad \vec{u_e} = \vec{u_0} + \vec{u_1} \qquad \vec{E} = \vec{E_0} + \vec{E_1} \tag{6.73}$$

Since we want to address an electrostatic wave, perturbation of $\vec{B}$ is not accounted for (*i.e.*, $\vec{B} = \vec{B_0}$). Before displacement of electrons, plasma is homogeneous and at rest, so that

$$\nabla n_0 = \vec{u_0} = \vec{E_0} = 0 \qquad \frac{\partial n_0}{\partial t} = 0 \tag{6.74}$$

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Using equations (6.73) and (6.74) in (6.70) – (6.72) and on linearisation, we get

$$m_e \frac{\partial \vec{u_1}}{\partial t} = -e(\vec{E_1} + \vec{u_1} \times \vec{B_0}) \tag{6.75}$$

$$\frac{\partial n_1}{\partial t} + n_0 \nabla . \vec{u_1} = 0 \tag{6.76}$$

$$\epsilon_0 \nabla . \vec{E_1} = -en_1 \tag{6.77}$$

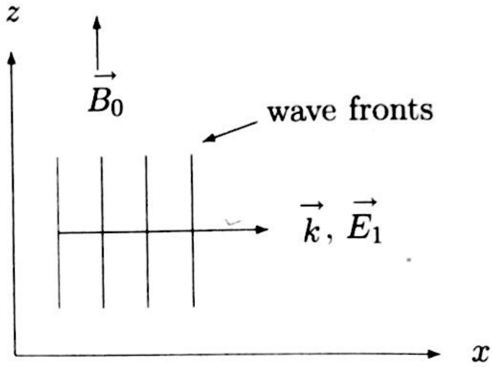

Figure 6.3: A longitudinal plane wave propagating at right angle to $\vec{B_0}$

Here, we shall consider only longitudinal waves with $\vec{k}$ parallel $\vec{E_1}$. Let us consider x-axis along $\vec{k}$ and $\vec{E_1}$, and z-axis along $\vec{B_0}$ (Figure 6.3). Hence, $\vec{k} = k\hat{i}$; $\vec{E_1} = E_1\hat{i}$; $\vec{B_0} = B_0\hat{k}$; $\vec{u_1} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k}$. We assume that the oscillating quantities behave sinusoidally as $e^{i(\vec{k}.\vec{r}-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by $i\vec{k}$. From the equations (6.75) – (6.77), we get

$$-im_e\omega u_x = -eE_1 - eu_yB_0 \tag{6.78}$$

$$-im_e\omega u_y = eu_xB_0 \tag{6.79}$$

$$-im_e\omega u_z = 0 \tag{6.80}$$

$$-i\omega n_1 + n_0 iku_x = 0 \tag{6.81}$$

$$ik\epsilon_0 E_1 = -en_1 \tag{6.82}$$

Using u_y from equation (6.79) in (6.78), we get

$$-im_e\omega u_x = -eE_1 - eB_0\Big(\frac{eB_0}{-im_e\omega}\Big)u_x$$

Thus,

$$u_x = \frac{eE_1}{im_e\omega} + \frac{e^2B_0^2}{\omega^2 m_e^2}u_x = \frac{eE_1}{im_e\omega} + \frac{\omega_c^2}{\omega^2}u_x$$

giving,

$$u_x = \frac{eE_1}{im_e\omega}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} \tag{6.83}$$

where $\omega_c = eB_0/m_e$ is the cyclotron frequency. u_x obviously tends to infinite when $\omega = \omega_c$. Equations (6.81) – (6.83) can be rearranged as

$$ik\epsilon_0 E_1 = -e\left(\frac{n_0 k}{\omega}\right)u_x$$

$$= -\left(\frac{n_0 ek}{\omega}\right)\frac{eE_1}{im_e\omega}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1}$$

so that

$$\left[1 - \frac{\omega_c^2}{\omega^2} - \frac{n_0\ e^2}{\epsilon_0\ m_e}\frac{1}{\omega^2}\right]E_1 = 0$$

Since E_1 is not zero, we have

$$1 - \frac{\omega_c^2}{\omega^2} = \frac{n_0\ e^2}{\epsilon_0\ m_e}\frac{1}{\omega^2} = \frac{\omega_p^2}{\omega^2}$$

where $\omega_p = (n_0 e^2/\epsilon_0 m_e)^{1/2}$ is the plasma frequency. Thus, we have

$$\omega^2 = \omega_p^2 + \omega_c^2 \equiv \omega_h^2 \tag{6.84}$$

The frequency ω_h is known as the *upper hybrid frequency*. Electrostatic electron waves perpendicular to $\vec{B}$ have this frequency ω_h whereas the electron waves parallel to $\vec{B}$ have the frequency ω_p. The group velocity is zero, as thermal motion is not accounted for. As the magnetic field goes to zero, ω_c goes to zero and we have the plasma oscillations with frequency ω_p. As the electron density goes to zero, ω_p goes to zero and we have a simple Larmor gyration. Existence of the upper hybrid frequency has been verified experimentally.

Equation (6.84) can be written as

$$\frac{\omega_c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} = 1 - \frac{n_0 e^2}{\epsilon_0 m_e \omega^2}$$

It shows that ω_c^2/ω^2 varies linearly with the density n_0.

6.11 Electrostatic ion waves perpendicular to $\vec{B}$

Let us now investigate the ion acoustic wave where $\vec{k}$ is perpendicular to $\vec{B_0}$. Here, we consider that the angle between $\vec{k}$ and $\vec{B_0}$ is almost $\pi/2$ so that $\vec{k} \cdot \vec{B_0} \neq 0$. The geometry is shown in Figure 6.4.

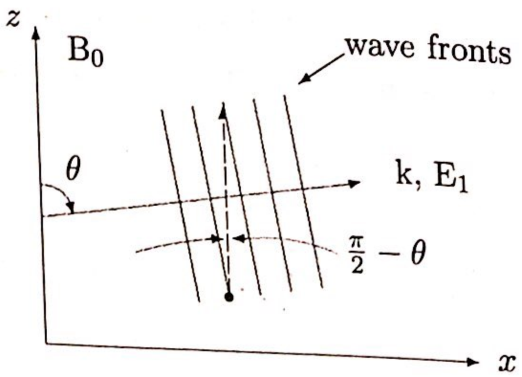

Figure 6.4: An electrostatic ion cyclotron wave propagating nearly at right angle to $\vec{B_0}$

We also consider electrostatic waves with $\vec{B_1} = 0$, and therefore, $\vec{k} \times \vec{E_1} = 0$, *i.e.*, $\nabla \times \vec{E_1} = 0$. Therefore, $\vec{E_1} = -\nabla\phi$, where ϕ is a scalar quantity. We assume also that there are no thermal motions ($T_i = 0$). The equations of motion and continuity for an ion are

$$m_i n_i \left[\frac{\partial \vec{u_i}}{\partial t} + (\vec{u_i} \cdot \nabla)\vec{u_i}\right] = e n_i(-\nabla\phi + \vec{u_i} \times \vec{B_0}) \tag{6.85}$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{u_i}) = 0 \tag{6.86}$$

Here, we have used the plasma approximation $n_i = n_e = n$, and therefore, we shall not use Poisson equation. When amplitude of each oscillating quantity is small, let us linearize the equations by neglecting the terms having two and higher powers of amplitude. For this, let us first separate the variable parameters into two parts: (i) an equilibrium part, which represents the state of plasma in absence of oscillations; and is indicated by a subscript 0 and (ii) the perturbation part due to oscillations, indicated by a subscript 1:

$$\vec{u_i} = \vec{u_0} + \vec{u_1} \qquad \phi = \phi_0 + \phi_1 \qquad n_i = n_0 + n_1 \tag{6.87}$$

Before displacement of electrons, plasma is homogeneous and at rest, so that

$$\nabla n_0 = \vec{u_0} = \nabla\phi_0 = 0 \qquad \frac{\partial n_0}{\partial t} = 0 \tag{6.88}$$

Using equations (6.87) and (6.88) in (6.85) and on linearisation, we get

$$m_i \frac{\partial \vec{u_1}}{\partial t} = -e\nabla\phi_1 + e\,\vec{u_1} \times \vec{B_0} \tag{6.89}$$

Assuming that electrons can move along $\vec{B_0}$. Fluid drift parallel to the magnetic field is the same as in its absence. For the electric field $\vec{E} = -\nabla\phi = -\nabla(\phi_0 + \phi_1) = -\nabla\phi_1$, the density of massless electrons is (mass of an electron may be considered as negligible in comparison to that of proton)

$$n_e = n = n_0 \exp\Big(\frac{e\phi_1}{KT_e}\Big)$$

$$= n_0\Big(1 + \frac{e\phi_1}{KT_e} + \ldots\Big) = n_0\Big(1 + \frac{e\phi_1}{KT_e}\Big)$$

Neglecting higher order terms, we have

$$n - n_0 = n_0\,\frac{e\phi_1}{KT_e} \qquad n_1 = n_0\,\frac{e\phi_1}{KT_e} \tag{6.90}$$

Using equations (6.87) and (6.88) in (6.86) and on linearisation, we get

$$\frac{\partial n_1}{\partial t} + n_0\nabla.\,\vec{u_1} = 0 \tag{6.91}$$

Here, we shall consider only longitudinal waves with $\vec{k}$ parallel $\vec{E_1}$. Let us consider x-axis along $\vec{k}$ and $\vec{E_1}$, and z-axis along $\vec{B_0}$. Further, let the oscillating quantities behave sinusoidally. Hence, $\vec{k} = k\hat{i}$; $\vec{B_0} = B_0\hat{k}$; $\vec{u_1} = (u_x\hat{i} + u_y\hat{j} + u_z\hat{k})$. We assume that the oscillating quantities behave sinusoidally as $e^{i(\vec{k}.\vec{r}-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by $i\,\vec{k}$. From the equations (6.89) – (6.91), we get

$$-i\omega m_i u_x = -eik\phi_1 + eu_yB_0 \tag{6.92}$$

$$-i\omega m_i u_y = -eu_xB_0 \tag{6.93}$$

$$-i\omega m_i u_z = 0 \tag{6.94}$$

$$n_1 = n_0\frac{e\phi_1}{KT_e} \tag{6.95}$$

$$-i\omega n_1 + n_0iku_x = 0 \tag{6.96}$$

Using u_y from equation (6.93) in (6.92), we get

$$-i\omega m_i u_x = -eik\phi_1 + eB_0\Big(\frac{eB_0}{i\omega m_i}\Big)u_x$$

Hence,

$$u_x = \frac{ek\phi_1}{\omega m_i} + \Big(\frac{e^2B_0^2}{\omega^2 m_i^2}\Big)u_x = \frac{ek\phi_1}{\omega m_i} + \frac{\Omega_c^2}{\omega^2}u_x$$

giving,

$$u_x = \frac{ek\phi_1}{\omega m_i}\Big(1-\frac{\Omega_c^2}{\omega^2}\Big)^{-1} \tag{6.97}$$

where $\Omega_c = eB_0/m_i$ is the ion cyclotron frequency. Equations (6.95) – (6.97) can be rearranged as

$$u_x = \frac{ek}{\omega m_i}\Big(1-\frac{\Omega_c^2}{\omega^2}\Big)^{-1}\Big(\frac{KT_e}{n_0 e}\Big)n_1$$

$$= \frac{ek}{\omega m_i}\Big(1-\frac{\Omega_c^2}{\omega^2}\Big)^{-1}\Big(\frac{KT_e}{n_0 e}\Big)\Big(\frac{n_0 k}{\omega}\Big)u_x$$

Since u_x is not zero, we have

$$\omega^2 - \Omega_c^2 = k^2\frac{KT_e}{m_i}$$

Since we have assumed $T_i = 0$, we can write

$$\omega^2 = \Omega_c^2 + k^2 v_s^2$$

This is dispersion relation for electrostatic ion waves. Electrostatic ion waves also have been detected experimentally.

6.12 Lower hybrid frequency

In the preceding section, we considered that the angle between $\vec{k}$ and $\vec{B}_0$ is almost $\pi/2$ so that $\vec{k}\cdot\vec{B}_0 \neq 0$. When the angle between $\vec{k}$ and $\vec{B}_0$ is exactly $\pi/2$, the electrons are not allowed to preserve neutrality by flowing along the lines of force, and hence they also obey equation of motion. The equations of motion and continuity for an ion and an electron are

$$m_i n_i\Big[\frac{\partial \vec{u}_i}{\partial t} + (\vec{u}_i\,.\nabla)\,\vec{u}_i\Big] = en_i(-\nabla\phi + \vec{u}_i\times\vec{B}_0) \tag{6.98}$$

$$\frac{\partial n_i}{\partial t} + \nabla .\ (n_i\ \vec{u_i}) = 0 \tag{6.99}$$

$$m_e n_e \Big[\frac{\partial \vec{v_e}}{\partial t} + (\ \vec{v_e}\ .\nabla)\ \vec{u_i}\ \Big] = -en_e(-\nabla\phi + \vec{v_e} \times \vec{B_0}) \tag{6.100}$$

$$\frac{\partial n_e}{\partial t} + \nabla .\ (n_e\ \ \vec{u_i}) = 0 \tag{6.101}$$

Equations (6.98) and (6.99) are solved in the preceding section and we have

$$u_{ix} = \frac{ek\phi_1}{\omega m_i}\Big(1 - \frac{\Omega_c^2}{\omega^2}\Big)^{-1} \qquad \text{and} \qquad n_{i1} = \frac{n_0 k}{\omega} u_{ix} \tag{6.102}$$

Solution of the equations (6.100) and (6.101) can be obtained in a similar manner by replacing e by $-e$, m_i by m_e, Ω_c by $-\omega_c$ in equations (6.102) as

$$u_{ex} = -\frac{ek\phi_1}{\omega m_e}\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{-1} \qquad \text{and} \qquad n_{e1} = \frac{n_0 k}{\omega} u_{ex} \tag{6.103}$$

The plasma approximation $n_i = n_e = n_0$ requires $n_{i1} = n_{e1}$ and thus, $u_{ix} = u_{ex}$ (using second equations of 6.102 and 6.103). From equations (6.102) and (6.103), we have

$$m_i\Big(1 - \frac{\Omega_c^2}{\omega^2}\Big) = -m_e\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)$$

$$\omega^2(m_i + m_e) = m_e\omega_c^2 + m_i\Omega_c^2$$

$$= e^2 B_0^2\Big(\frac{1}{m_i} + \frac{1}{m_e}\Big)$$

giving,

$$\omega^2 = \frac{e^2 B_0^2}{m_i m_e} = \Omega_c\omega_c \qquad\qquad \omega = \sqrt{\Omega_c\omega_c} \equiv \omega_l$$

This frequency ω_l is known as the *lower hybrid frequency.*

6.13 Electromagnetic waves with $\vec{B_0}= 0$

In a plasma, we have $\vec{E_0} = 0$ and $\vec{j_0} = 0$. Thus, for $\vec{B_0} = 0$, let us consider the Maxwell equations

$$\nabla\times \vec{E_1}= -\frac{\partial \vec{B_1}}{\partial t} \tag{6.104}$$

$$\nabla\times \vec{B_1}= \mu_0\Big(\vec{j_1} +\epsilon_0\frac{\partial \vec{E_1}}{\partial t}\Big) \tag{6.105}$$

Equation (6.105) can be rearranged as

$$c^2\nabla\times \vec{B_1}= \frac{\vec{j_1}}{\epsilon_0} + \frac{\partial \vec{E_1}}{\partial t} \tag{6.106}$$

$c = 1/\sqrt{\mu_0\epsilon_0}$ the speed of light. Differentiating equation (6.106) with respect to time, we have

$$c^2\nabla\times \frac{\partial \vec{B_1}}{\partial t} = \frac{1}{\epsilon_0}\frac{\partial \vec{j_1}}{\partial t} + \frac{\partial^2 \vec{E_1}}{\partial t^2} \tag{6.107}$$

Taking curl of equation (6.104), we have

$$\nabla\times(\nabla\times \vec{E_1}) = -\nabla\times \frac{\partial \vec{B_1}}{\partial t}$$

$$\nabla(\nabla.\,\vec{E_1}) - \nabla^2\vec{E_1}= -\nabla\times \frac{\partial \vec{B_1}}{\partial t} \tag{6.108}$$

Using equation (6.107) in equation (6.108), we have

$$\nabla(\nabla.\,\vec{E_1}) - \nabla^2\vec{E_1}= -\frac{1}{c^2\epsilon_0}\frac{\partial \vec{j_1}}{\partial t} - \frac{1}{c^2}\frac{\partial^2 \vec{E_1}}{\partial t^2} \tag{6.109}$$

We assume that the oscillating quantities behave sinusoidally as $e^{i(\vec{k}.\vec{r}-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by $i\vec{k}$. From the equation (6.109), we get

$$i\vec{k}\,(i\vec{k}\,.\,\vec{E_1}) - (i\vec{k})^2\,\vec{E_1}= -\frac{-i\omega}{c^2\epsilon_0}\,\vec{j_1} - \frac{(-i\omega)^2}{c^2}\,\vec{E_1}$$

$$-\vec{k}\,(\vec{k}\,.\,\vec{E_1}) + k^2\,\vec{E_1}= \frac{i\omega}{c^2\epsilon_0}\,\vec{j_1} + \frac{\omega^2}{c^2}\,\vec{E_1}$$

For the transverse wave, we have $\vec{k} \cdot \vec{E_1} = 0$. Thus,

$$c^2k^2 \vec{E_1} = \frac{i\omega}{\epsilon_0} \vec{j_1} + \omega^2 \vec{E_1} \qquad (\omega^2 - c^2k^2) \vec{E_1} = -\frac{i\omega}{\epsilon_0} \vec{j_1} \quad (6.110)$$

For high frequency waves (*e.g.*, light waves and microwaves), the ions may be considered as fixed. Then the current density is due to the motion of electrons only is

$$\vec{j_1} = -n_0 e \vec{u}_{e1}$$

The equation of motion is

$$m_e n_e \left[\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e . \nabla) \vec{u}_e \right] = -e n_e (\vec{E} + \vec{u}_e \times \vec{B})$$

Here, we are considering that the electron temperature T_e is zero. After linearisation with $\vec{u_0} = 0$, $\vec{E_0} = 0$ and $\vec{B_0} = 0$, we have

$$m_e \frac{\partial \vec{u}_{e1}}{\partial t} = -e \vec{E_1}$$

We assume that the oscillating quantities behave sinusoidally as $e^{i(\vec{k} \cdot \vec{r} - \omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$ and we get

$$m_e(-i\omega) \vec{u}_{e1} = -e\vec{E_1} \qquad \vec{u}_{e1} = \left(\frac{e}{i m_e \omega}\right) \vec{E_1} \quad (6.111)$$

Thus, we have

$$\vec{j_1} = -\left(\frac{n_0 e^2}{i m_e \omega}\right) \vec{E_1} \qquad (\omega^2 - c^2k^2) \vec{E_1} = -\frac{i\omega}{\epsilon_0} \left(- \frac{n_0 e^2}{i m_e \omega}\right) \vec{E_1}$$

Here we have used equation (6.110). Since $\vec{E_1} \neq 0$, we have

$$\omega^2 - c^2k^2 = \frac{n_0 e^2}{\epsilon_0 m_e} \qquad \omega^2 - c^2k^2 = \omega_p^2 \qquad \omega^2 = \omega_p^2 + c^2k^2$$

where $\omega_p = (n_0 e^2/\epsilon_0 m_e)^{1/2}$ is the plasma frequency. This is the dispersion relation for transverse electromagnetic waves propagating in a plasma with $\vec{B_0} = 0$. For electromagnetic waves in vacuum we have[2]

[2]Maxwell's equations for electromagnetic waves in vacuum are

$$\nabla \times \vec{E_1} = -\frac{\partial \vec{B_1}}{\partial t} \quad (6.112)$$

$\omega^2 = c^2k^2$. It shows that there is an additional term ω_p^2. The phase velocity is

$$\frac{\omega^2}{k^2} = \frac{\omega_p^2}{k^2} + c^2 \qquad \frac{\omega}{k} = c\sqrt{1 + \frac{\omega_p^2}{c^2k^2}} \equiv u_\phi$$

It shows that the phase velocity is always greater than c. The group velocity is

$$2\omega\ \mathrm{d}\omega = 2c^2k\ \mathrm{d}k$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c^2}{\omega/k} = \frac{c^2}{v_\phi} = c\left(1 + \frac{\omega_p^2}{c^2\ k^2}\right)^{-1/2} \equiv u_g$$

$$c^2\nabla\times \vec{B}_1 = \frac{\partial \vec{E}_1}{\partial t} \tag{6.113}$$

Since in a vacuum we have $\vec{j} = 0$, and $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light. Differentiating equation (6.112) with respect to time, we have

$$\nabla \times \frac{\partial \vec{E}_1}{\partial t} = -\frac{\partial^2 \vec{B}_1}{\partial t^2} \tag{6.114}$$

Taking curl of equation (6.113), we have

$$c^2\nabla \times (\nabla\times \vec{B}_1) = \nabla \times \frac{\partial \vec{E}_1}{\partial t} \tag{6.115}$$

Using equation (6.114) in equation (6.115), we have

$$c^2\nabla \times (\nabla\times \vec{B}_1) = -\frac{\partial^2 \vec{B}_1}{\partial t^2}$$

$$c^2\left[\nabla(\nabla.\ \vec{B}_1) - (\nabla.\nabla)\ \vec{B}_1)\right] = -\frac{\partial^2 \vec{B}_1}{\partial t^2}$$

$$-c^2(\nabla.\nabla)\ \vec{B}_1) = -\frac{\partial^2 \vec{B}_1}{\partial t^2}$$

where we have used $\nabla.\vec{B}_1 = 0$. We assume that the oscillating quantities behave sinusoidally as $\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega t)}$. Then time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by $i\vec{k}$, and we have

$$c^2(\vec{k}\ .\ \vec{k})\ \vec{B}_1 = \omega^2\ \vec{B}_1 \qquad c^2k^2\ \vec{B}_1 = \omega^2\ \vec{B}_1$$

Since $\vec{B}_1 \neq 0$, we have

$$c^2k^2 = \omega^2 \qquad \frac{\omega}{k} = c \qquad \frac{\mathrm{d}\omega}{\mathrm{d}k} = c$$

Thus, phase velocity as well as group velocity of the light waves is c.

It shows that the group velocity is always smaller than c. When $k \to \infty$, the phase velocity $u_\phi \to c$, and also the group velocity $u_\phi \to c$. This dispersion relation is shown in Figure 6.5.

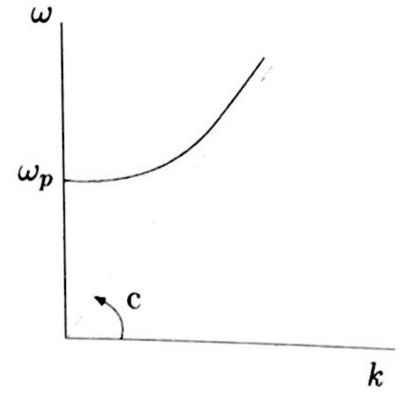

Figure 6.5: Variation of ω_p with k for electromagnetic waves in a plasma with no DC magnetic field

This figure is similar to that in the case of the electron plasma waves, but the behaviour is quite different as the asymptotic velocity in the present case is c and in that case was $\sqrt{3/2}\; v_{th}$. Further, there is a difference in the damping as in the case of electron waves

$$k = \sqrt{\frac{2}{3}}\frac{1}{v_h}\sqrt{\omega^2 - \omega_p^2}$$

is larger than the present value

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}$$

At the critical density $n = n_c$, we have $\omega = \omega_p$. When $\omega < \omega_p$, the wave vector k is an imaginary number. Since spatial dependence of the wave is $\exp(ikx)$, it will be attenuated for imaginary value of k. The skin depth δ is obtained as

$$e^{ikx} = e^{-|k|x} = e^{-k/\delta} \qquad\qquad \delta = \frac{1}{|k|} = \frac{c}{(\omega_p^2 - \omega^2)^{1/2}}$$

Thus, there is a cut-off frequency for a wave to propagate through the plasma. The phenomenon of cut-off frequency is an easy way to measure plasma density. With the increase of the density, ω_p increases and the frequencies satisfying $\omega < \omega_p$ are attenuated. Attenuation of a wave

with frequency ω shows that the density in the region is more than the critical value

$$n_c = \frac{m_e \epsilon_0 \omega^2}{e^2}$$

Exercise 1: CO_2 laser operates at $\lambda = 10.6\ \mu$ m. Calculate the minimum cut-off density of the plasma that does not allow transmission of the laser beam.

Solution: The frequency of the CO_2 laser is

$$f = \frac{c}{\lambda} = \frac{3 \times 10^8}{10.6 \times 10^{-6}} = 2.8 \times 10^{13} \text{ Hz}$$

The critical density is

$$n_c = \frac{m_e \epsilon_0 (2\pi f)^2}{e^2}$$

$$= \frac{9.1 \times 10^{-31} \times 8.854 \times 10^{-12} \times (2 \times 3.14 \times 2.8 \times 10^{13})^2}{(1.6 \times 10{-}19)^2}$$

$$= 10^{25} \text{m}^{-3}$$

Thus, the minimum cut-off density is 10^{25}m^{-3}.

6.14 Electromagnetic waves perpendicular to $\vec{B}_0$

In the last section, we considered propagation of electromagnetic waves when there is no magnetic field. Now, we account for the propagation of electromagnetic waves when a magnetic field $\vec{B}_0$ is present. In the present section, we consider the case of perpendicular propagation, $\vec{k} \perp \vec{B}_0$. If we take the transverse waves with $\vec{k} \perp \vec{E}_1$, we have two options for the direction of $\vec{E}_1$: (i) $\vec{E}_1$ may be parallel to $\vec{B}_0$ – the case of ordinary waves and (ii) $\vec{E}_1$ may be perpendicular to $\vec{B}_0$ – the case of extraordinary waves. Let us consider these cases one by one.

I. Ordinary waves – the case of $\vec{E}_1 \| \vec{B}_0$

In a plasma, we have $\vec{E}_0 = 0$ and $\vec{j}_0 = 0$. Let us consider $\vec{B}_0 = B_0\hat{k}$, $\vec{k} = k\hat{i}$, and $\vec{E}_1 = E_1\hat{k}$. Since the magnetic field $\vec{B}_0$ is uniform and

independent of time, we still have the Maxwell equations

$$\nabla\times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} \qquad \text{and} \qquad c^2\nabla\times \vec{B}_1 = \frac{1}{\epsilon_0}\vec{j}_1 + \frac{\partial \vec{E}_1}{\partial t} \tag{6.116}$$

where $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light. As discussed in the preceding section, equation (6.116) can be transformed to

$$-\vec{k}\,(\vec{k}\cdot\vec{E}_1) + k^2\,\vec{E}_1 = \frac{i\omega}{c^2\epsilon_0}\,\vec{j}_1 + \frac{\omega^2}{c^2}\,\vec{E}_1$$

For the transverse wave, we have $\vec{k}\cdot\vec{E}_1 = 0$ and thus,

$$c^2k^2\vec{E}_1 = \frac{i\omega}{\epsilon_0}\,\vec{j}_1 + \omega^2\,\vec{E}_1 \qquad (\omega^2 - c^2k^2)\,\vec{E}_1 = -\frac{i\omega}{\epsilon_0}\,\vec{j}_1 \tag{6.117}$$

For high frequency waves (*e.g.*, light waves and microwaves), the ions may be considered as fixed. Then the current density is due to the motion of electrons as

$$\vec{j}_1 = -n_0 e\vec{u}_{e1} \tag{6.118}$$

The equation of motion is

$$m_e n_e\left[\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e\cdot\nabla)\vec{u}_e\right] = -en_e(\vec{E} + \vec{u}_e\times\vec{B})$$

Here, we are assuming that the electron temperature T_e is zero. After linearisation with $\vec{u_0} = 0$, and $\vec{E}_0 = 0$, we have

$$m_e\frac{\partial \vec{u}_{e1}}{\partial t} = -e\vec{E}_1 - e\vec{u}_{e1}\times\vec{B}_0$$

Since $\vec{E}_1$ is parallel to z-axis, we are interested in the z-component only, therefore, the term $\vec{u}_{e1}\times\vec{B}_0$ does not contribute. We assume that the oscillating quantities behave sinusoidally as $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$ and on linearisation we get

$$m_e(-i\omega)\vec{u}_{e1} = -e\vec{E}_1 \qquad \vec{u}_{e1} = \left(\frac{e}{im_e\omega}\right)\vec{E}_1 \tag{6.119}$$

From equations (6.118) and (6.119), we have

$$\vec{j}_1 = -\left(\frac{n_0e^2}{im_e\omega}\right)\vec{E}_1$$

Using this equation in (6.117), we have

$$(\omega^2 - c^2k^2)\,\vec{E_1} = -\frac{i\omega}{\epsilon_0}\Big(-\frac{n_0e^2}{im_e\omega}\Big)\vec{E_1}$$

Since $\vec{E_1} \neq 0$, we have

$$\omega^2 - c^2k^2 = \frac{n_0e^2}{\epsilon_0 m_e} \qquad\qquad \omega^2 = \omega_p^2 + c^2k^2$$

where $\omega_p = (n_0e^2/\epsilon_0 m_e)^{1/2}$ is the plasma frequency. This is the dispersion relation for electromagnetic waves propagating perpendicular to $\vec{B_0}$.

Notice that this relation is the same as for the transverse electromagnetic waves in absence of the magnetic field, and therefore these waves are known as the *ordinary waves.* The reason for that is that $\vec{B_0}$ being parallel to $\vec{E_1}$ does not affect the motion of electrons. The phase velocity is

$$\frac{\omega^2}{k^2} = \frac{\omega_p^2}{k^2} + c^2 \qquad\qquad \frac{\omega}{k} = c\sqrt{1 + \frac{\omega_p^2}{c^2\,k^2}} \equiv v_\phi$$

It shows that the phase velocity is always greater than c. The group velocity is

$$2\omega\ \mathrm{d}\omega = 2c^2k\ \mathrm{d}k \qquad\qquad \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c^2}{v_\phi} = c\Big(1 + \frac{\omega_p^2}{c^2k^2}\Big)^{-1/2} \equiv v_g$$

It shows that the group velocity is always smaller than c. When $k \to \infty$, the phase velocity $v_\phi \to c$, and thus, the group velocity $v_\phi \to c$. This dispersion relation is shown in Figure 6.5. This figure is similar to that in the case of the electron plasma waves, but the behaviour is quite different as the asymptotic velocity in the present case is c and in that case was $\sqrt{3/2}\ v_{th}$. Further, there is a difference in the damping as in the case of electron waves, we have

$$k = \sqrt{\frac{2}{3}}\frac{1}{v_h}\sqrt{\omega^2 - \omega_p^2}$$

which is larger than the present value

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}$$

At the critical density $n = n_c$, we have $\omega = \omega_p$. When $\omega < \omega_p$, the wave vector k is an imaginary number. Since spatial dependence of the wave is $\exp(ikx)$, it will be attenuated for imaginary value of k. The skin depth δ is obtained as

$$e^{ikx} = e^{-|k|x} = e^{-k/\delta} \qquad \delta = \frac{1}{|k|} = \frac{c}{(\omega_p^2 - \omega^2)^{1/2}}$$

Thus, there is a cut-off frequency for a wave to propagate through the plasma. The phenomenon is cut-off frequency is an easy way to measure plasma density. With the increase of the density, ω_p increases and the frequencies satisfying $\omega < \omega_p$ are attenuated. Attenuation of a wave with frequency ω shows that the density in the region is more than the critical value

$$n_c = \frac{m_e \epsilon_0 \omega^2}{e^2}$$

II. Extraordinary waves – the case of $\vec{E}_1 \perp \vec{B}_0$

Here, $\vec{B}_0$ being perpendicular to $\vec{E}_1$ affects the motion of electrons. Let us consider $\vec{B}_0 = B_0\hat{k}$, $\vec{k} = k\hat{i}$, and $\vec{E}_1 = E_x\hat{i} + E_y\hat{j}$. In a plasma, we have $\vec{E}_0 = 0$ and $\vec{j}_0 = 0$. Since the magnetic field $\vec{B}_0$ is uniform and independent of time, we still have the Maxwell equations

$$\nabla\times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} \qquad \text{and} \qquad c^2\nabla\times \vec{B}_1 = \frac{1}{\epsilon_0}\vec{j}_1 + \frac{\partial \vec{E}_1}{\partial t} \tag{6.120}$$

where $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light. As discussed in the preceding section, equation (6.120) can be transformed to

$$-\vec{k}\,(\vec{k}.\vec{E}_1) + k^2\vec{E}_1 = \frac{i\omega}{c^2\epsilon_0}\vec{j}_1 + \frac{\omega^2}{c^2}\vec{E}_1$$

Now, we have $\vec{k}.\vec{E}_1 = kE_x$. Thus,

$$-c^2kE_x\vec{k} + c^2k^2\vec{E}_1 = \frac{i\omega}{\epsilon_0}\vec{j}_1 + \omega^2\vec{E}_1$$

$$(\omega^2 - c^2k^2)\vec{E}_1 + c^2kE_x\vec{k} = -\frac{i\omega}{\epsilon_0}\vec{j}_1 \tag{6.121}$$

For high frequency waves (*e.g.*, light waves and microwaves), the ions may be considered as fixed. Then the current density is due to the motion of electrons as

$$\vec{j}_1 = -n_0 e \vec{u}_{e1} \tag{6.122}$$

Using equation (6.122) in (6.121), we have

$$(\omega^2 - c^2k^2)\vec{E}_1 + c^2 k E_x \vec{k} = \frac{i n_0 \omega e}{\epsilon_0} \vec{u}_{e1}$$

.

The x and y components of this equation are

$$(\omega^2 - c^2k^2)E_x + c^2k^2E_x = \frac{i n_0 \omega e}{\epsilon_0} u_x \qquad \omega^2 E_x = \frac{i n_0 \omega e}{\epsilon_0} u_x \tag{6.123}$$

and

$$(\omega^2 - c^2k^2)E_y = \frac{i n_0 \omega e}{\epsilon_0} u_y \tag{6.124}$$

The equation of motion is

$$m_e n_e \left[\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e . \nabla)\vec{u}_e \right] = -e n_e (\vec{E} + \vec{u}_e \times \vec{B})$$

Here, we are assuming that the electron temperature T_e is zero. After linearisation with $\vec{u_0} = 0$ and $\vec{E_0} = 0$, we have

$$m_e \frac{\partial \vec{u}_{e1}}{\partial t} = -e(\vec{E_1} + \vec{u}_{e1} \times \vec{B_0})$$

We assume that the oscillating quantities behave sinusoidally as $e^{i(\vec{k}.\vec{r} - \omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$ and we get

$$m_e(-i\omega)\vec{u}_{e1} = -e(\vec{E_1} + \vec{u}_{e1} \times \vec{B_0})$$

Considering for non-trivial x and y components, we have

$$-i m_e \omega u_x = -e(E_x + u_y B_0) \qquad \text{and} \qquad -i m_e \omega u_y = -e(E_y - u_x B_0)$$

Using $\omega_c = eB_0/m_e$, we have

$$u_x = -i\frac{e}{m_e\omega}\Big(E_x + u_y B_0\Big) = -\frac{i\omega_c}{\omega}\Big(\frac{E_x}{B_0} + u_y\Big) \tag{6.125}$$

and

$$u_y = -i\frac{e}{m_e\omega}\Big(E_y - u_x B_0\Big) = -\frac{i\omega_c}{\omega}\Big(\frac{E_y}{B_0} - u_x\Big) \tag{6.126}$$

Using equation (6.126) in (6.125), we have

$$u_x = -\frac{i\omega_c}{\omega}\Big[\frac{E_x}{B_0} - \frac{i\omega_c}{\omega}\Big(\frac{E_y}{B_0} - u_x\Big)\Big] = -\frac{i\omega_c}{\omega}\frac{E_x}{B_0} - \frac{\omega_c^2}{\omega^2}\frac{E_y}{B_0} + \frac{\omega_c^2}{\omega^2}u_x$$

$$u_x\Big(1 - \frac{\omega_c^2}{\omega^2}\Big) = \frac{\omega_c}{\omega B_0}\Big[-iE_x - \frac{\omega_c}{\omega}E_y\Big] = \frac{e}{m_e\omega}\Big[-iE_x - \frac{\omega_c}{\omega}E_y\Big]$$

$$u_x = \frac{e}{m_e\omega}\Big[-iE_x - \frac{\omega_c}{\omega}E_y\Big]\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{-1} \tag{6.127}$$

Using equation (6.125) in (6.126), we have

$$u_y = -\frac{i\omega_c}{\omega}\Big[\frac{E_y}{B_0} + \frac{i\omega_c}{\omega}\Big(\frac{E_x}{B_0} + u_y\Big)\Big] = -\frac{i\omega_c}{\omega}\frac{E_y}{B_0} + \frac{\omega_c^2}{\omega^2}\frac{E_x}{B_0} + \frac{\omega_c^2}{\omega^2}u_y$$

$$u_y\Big(1 - \frac{\omega_c^2}{\omega^2}\Big) = \frac{\omega_c}{\omega B_0}\Big[-iE_y + \frac{\omega_c}{\omega}E_x\Big] = \frac{e}{m_e\omega}\Big[-iE_y + \frac{\omega_c}{\omega}E_x\Big]$$

$$u_y = \frac{e}{m_e\omega}\Big[-iE_y + \frac{\omega_c}{\omega}E_x\Big]\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{-1} \tag{6.128}$$

Using equation (6.127) in (6.123), we have

$$\omega^2 E_x = -\frac{in_0\omega e}{\epsilon_0}\frac{e}{m_e\omega}\Big[iE_x + \frac{\omega_c}{\omega}E_y\Big]\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{-1}$$

$$\omega^2\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)E_x = -i\omega_p^2\Big[iE_x + \frac{\omega_c}{\omega}E_y\Big] = \omega_p^2 E_x - \frac{i\omega_p^2\omega_c}{\omega}E_y$$

$$(\omega^2 - \omega_h^2)E_x + \frac{i\omega_p^2\omega_c}{\omega}E_y = 0 \tag{6.129}$$

where $\omega_h = \sqrt{\omega_c^2 + \omega_p^2}$ is the upper hybrid frequency. Using equation (6.127) in (6.128), we have

$$(\omega^2 - c^2k^2)E_y = -\frac{in_0\omega e}{\epsilon_0}\frac{e}{m_e\omega}\Big[iE_y - \frac{\omega_c}{\omega}E_x\Big]\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{-1}$$

$$(\omega^2 - c^2k^2)\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)E_y = \omega_p^2 E_y + \frac{i\omega_p^2\omega_c}{\omega}E_x$$

$$-\frac{i\omega_p^2\omega_c}{\omega}E_x + \Big[(\omega^2 - c^2k^2)\Big(1 - \frac{\omega_c^2}{\omega^2}\Big) - \omega_p^2\Big]E_y = 0 \tag{6.130}$$

For convenience, let us express equations (6.129) and (6.130) as

$$AE_x + BE_y = 0 \qquad \text{and} \qquad CE_x + DE_y = 0$$

These simultaneous equations are satisfied when

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 0$$

$$AD - CB = 0 \qquad AD = CB$$

Using A, B, C and D, we get

$$(\omega^2 - \omega_h^2)\left[(\omega^2 - c^2k^2)\left(1 - \frac{\omega_c^2}{\omega^2}\right) - \omega_p^2\right] = \frac{i\omega_p^2\omega_c}{\omega}\left(-\frac{i\omega_p^2\omega_c}{\omega}\right)$$

$$(\omega^2 - \omega_h^2)\left[\omega^2 - \omega_h^2 - c^2k^2\left(1 - \frac{\omega_c^2}{\omega^2}\right)\right] = \frac{\omega_p^4\omega_c^2}{\omega^2}$$

$$c^2k^2\left(1 - \frac{\omega_c^2}{\omega^2}\right) = (\omega^2 - \omega_h^2) - \frac{\omega_p^4\omega_c^2/\omega^2}{\omega^2 - \omega_h^2}$$

Thus,

$$\frac{c^2k^2}{\omega^2} = \frac{(\omega^2 - \omega_h^2) - \left[(\omega_p^2\omega_c/\omega)^2/(\omega^2 - \omega_h^2)\right]}{\omega^2 - \omega_c^2} = \frac{(\omega^2 - \omega_h^2)^2 - (\omega_p^4\omega_c^2/\omega^2)}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)}$$

$$= \frac{(\omega^2 - \omega_h^2)(\omega^2 - \omega_c^2 - \omega_p^2) - (\omega_p^4\omega_c^2/\omega^2)}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)}$$

$$= \frac{(\omega^2 - \omega_h^2)(\omega^2 - \omega_c^2)}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)} - \frac{\omega_p^2(\omega^2 - \omega_h^2) + (\omega_p^4\omega_c^2/\omega^2)}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)}$$

$$= 1 - \frac{\omega_p^2(\omega^2 - \omega_h^2) + (\omega_p^4\omega_c^2/\omega^2)}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)}$$

$$= 1 - \frac{\omega_p^2}{\omega^2}\left[\frac{\omega^2(\omega^2 - \omega_c^2) - \omega_p^2(\omega^2 - \omega_c^2)}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_h^2)}\right] = 1 - \frac{\omega_p^2}{\omega^2}\left[\frac{\omega^2 - \omega_p^2}{(\omega^2 - \omega_h^2)}\right]$$

Therefore, we have

$$\frac{c^2\,k^2}{\omega^2} = \frac{c^2}{v_\phi^2} = 1 - \frac{\omega_p^2}{\omega^2}\left[\frac{\omega^2 - \omega_p^2}{(\omega^2 - \omega_h^2)}\right] \tag{6.131}$$

This is dispersion relation for the extra-ordinary waves. It is considerably complicated.

6.14.1 Cut-offs and resonances

In the dispersion relation for the extra-ordinary waves, one can find the cut-offs and the resonances. When the refractive index $\tilde{n} = c/v_\phi$ tends to zero, we have a cut-off. At the cut-off, the wavelength tends to infinite and the wave is reflected back. When the refractive index $\tilde{n}$ tends to infinite, we have a resonance. At the resonance, the wavelength tends to zero, and the wave is absorbed. When a wave propagates through the plasma in which ω_p and ω_c are varying, it may encounter cut-offs as well as resonances. Equation (6.131) shows that at $\omega = \omega_p$, we have $v_\phi = c$.

Resonances for the extra-ordinary waves can be obtained from equation (6.131) by putting $k = \infty$. It occurs when $\omega \to \omega_h$. Hence, at

$$\omega = \omega_h = \sqrt{\omega_p^2 + \omega_c^2}$$

the wave encounters a resonance. When the wave of frequency ω approaches the resonance point, its phase velocity as well as group velocity both approach to zero, and the wave energy is converted into upper hybrid oscillations.

Cut-offs for the extra-ordinary waves can be obtained from equation (6.131) by putting $k = 0$. Thus, we have

$$1 - \frac{\omega_p^2(\omega^2 - \omega_p^2)}{\omega^2(\omega^2 - \omega_h^2)} = 0; \qquad \frac{\omega_p^2}{\omega^2} = \frac{\omega^2 - \omega_p^2 - \omega_c^2}{\omega^2 - \omega_p^2}; \qquad 1 - \frac{\omega_p^2}{\omega^2} = \frac{\omega_c^2/\omega^2}{1 - \omega_p^2/\omega^2}$$

$$\left(1 - \frac{\omega_p^2}{\omega^2}\right)^2 = \frac{\omega_c^2}{\omega^2}; \qquad 1 - \frac{\omega_p^2}{\omega^2} = \pm\frac{\omega_c}{\omega} \qquad \omega^2 \mp \omega\omega_c - \omega_p^2 = 0$$

Thus, the cut-off frequencies are given by

$$\omega = \frac{\pm\omega_c \pm \sqrt{\omega_c^2 + 4\omega_p^4}}{2}$$

For the convention that ω is always positive, the roots are

$$\omega_R = \frac{\omega_c + \sqrt{\omega_c^2 + 4\omega_p^4}}{2} \qquad \text{and} \qquad \omega_L = \frac{-\omega_c + \sqrt{\omega_c^2 + 4\omega_p^4}}{2}$$

The cut-off frequencies ω_R and ω_L are known as the *right-hand* and *left-hand cutoffs*, respectively.

The cut-off and resonance frequencies divide the dispersion diagram into the regions of propagation and of non-propagation. Instead of the

ω versus k diagram, a plot of $\omega^2/c^2\ k^2$ versus ω is shown in Figure 6.6, where the wave does not propagate in the shaded region.

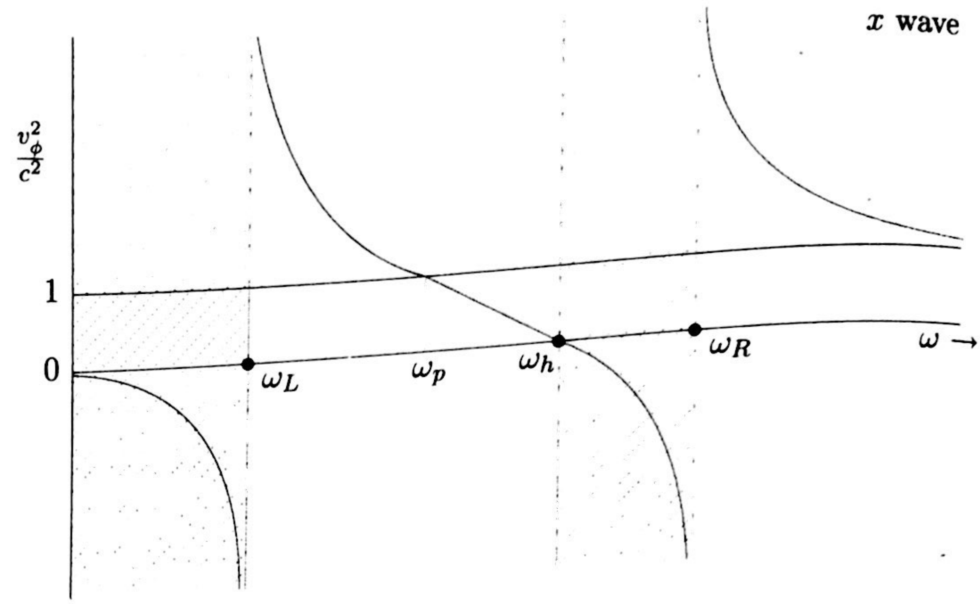

Figure 6.6: Variation of phase velocity with frequency for extraordinary wave. The wave does not propagate in the shaded regions.

6.15 Electromagnetic waves parallel to $\vec{B_0}$

In the last section, we considered the case of perpendicular propagation, $\vec{k}\perp\vec{B_0}$. Now, we take the case of $\vec{k}\|\vec{B_0}$. Let us consider $\vec{B_0} = B_0\hat{k}$, $\vec{k} = k\hat{k}$. For transverse propagation, we take $\vec{E_1} = E_x\hat{i} + E_y\hat{j}$. In a plasma, we have $\vec{E_0} = 0$ and $\vec{j_0} = 0$. Since the magnetic field $\vec{B_0}$ is uniform and independent of time, we still have the Maxwell equations

$$\nabla\times\vec{E_1} = -\frac{\partial\vec{B_1}}{\partial t} \qquad \text{and} \qquad c^2\nabla\times\vec{B_1} = \frac{1}{\epsilon_0}\vec{j_1} + \frac{\partial\vec{E_1}}{\partial t} \tag{6.132}$$

where $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light. As discussed earlier, equation (6.132) can be transformed to

$$-\vec{k}(\vec{k}.\vec{E_1}) + k^2\vec{E_1} = \frac{i\omega}{c^2\epsilon_0}\vec{j_1} + \frac{\omega^2}{c^2}\vec{E_1}$$

Now, we have $\vec{k}.\vec{E_1} = 0$. Thus, we have

$$c^2k^2\vec{E_1} = \frac{i\omega}{\epsilon_0}\vec{j_1} + \omega^2\vec{E_1} \qquad\qquad (\omega^2 - c^2k^2)\vec{E_1} = -\frac{i\omega}{\epsilon_0}\vec{j_1} \tag{6.133}$$

For high frequency waves (*e.g.*, light waves and microwaves), the ions may be considered as fixed. Then the current density is due to the motion of electrons as

$$\vec{j}_1 = -n_0 e \vec{u}_{e1} \tag{6.134}$$

Using equation (6.134) in (6.133), we have

$$(\omega^2 - c^2k^2)\vec{E}_1 = \frac{in_0\omega e}{\epsilon_0}\vec{u}_{e1} \tag{6.135}$$

The x and y components of this equation are

$$(\omega^2 - c^2k^2)E_x = \frac{in_0\omega e}{\epsilon_0}u_x \tag{6.136}$$

and

$$(\omega^2 - c^2k^2)E_y = \frac{in_0\omega e}{\epsilon_0}u_y \tag{6.137}$$

The equation of motion is

$$m_e n_e\Big[\frac{\partial \vec{u}_e}{\partial t} + (\vec{u}_e.\nabla)\vec{u}_e\Big] = -en_e(\vec{E} + \vec{u}_e \times \vec{B})$$

Here, we are assuming that the electron temperature T_e is zero. After linearisation with $\vec{u_0} = 0$, and $\vec{E_0} = 0$, we have

$$m_e\frac{\partial \vec{u}_{e1}}{\partial t} = -e(\vec{E_1} + \vec{u}_{e1} \times \vec{B_0})$$

We assume that the oscillating quantities behave sinusoidally as $e^{i(\vec{k}.\vec{r}-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$ and we get

$$m_e(-i\omega)\vec{u}_{e1} = -e(\vec{E_1} + \vec{u}_{e1} \times \vec{B_0})$$

Considering for non-trivial x and y components, we have

$$-im_e\omega u_x = -e(E_x + u_yB_0) \qquad \text{and} \qquad -im_e\omega u_y = -e(E_y - u_xB_0)$$

Using $\omega_c = eB_0/m_e$ here, we have

$$u_x = -i\frac{e}{m_e\omega}\Big(E_x + u_yB_0\Big) = -\frac{i\omega_c}{\omega}\Big(\frac{E_x}{B_0} + u_y\Big)$$

and

$$u_y = -i\frac{e}{m_e\omega}\Big(E_y - u_x B_0\Big) = -\frac{i\omega_c}{\omega}\Big(\frac{E_y}{B_0} - u_x\Big)$$

Solutions of these equations are (as discussed in the preceding section)

$$u_x = \frac{e}{m_e\omega}\Big[-iE_x - \frac{\omega_c}{\omega}E_y\Big]\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{-1} \tag{6.138}$$

and

$$u_y = \frac{e}{m_e\omega}\Big[-iE_y + \frac{\omega_c}{\omega}E_x\Big]\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{-1} \tag{6.139}$$

Using equations (6.138) and (6.139) in (6.136), we have

$$(\omega^2 - c^2k^2)E_x = -\frac{in_0\omega e}{\epsilon_0}\frac{e}{m_e\omega}\Big[iE_x + \frac{\omega_c}{\omega}E_y\Big]\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{-1}$$

$$(\omega^2 - c^2k^2)\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)E_x = -i\omega_p^2\Big[iE_x + \frac{\omega_c}{\omega}E_y\Big] = \omega_p^2E_x - \frac{i\omega_p^2\omega_c}{\omega}E_y$$

$$\Big[(\omega^2 - c^2k^2)\Big(1 - \frac{\omega_c^2}{\omega^2}\Big) - \omega_p^2\Big]E_x + \frac{i\omega_p^2\omega_c}{\omega}E_y = 0 \tag{6.140}$$

Using equations (6.138) and (6.139) in (6.137), we have

$$(\omega^2 - c^2k^2)E_y = -\frac{in_0\omega e}{\epsilon_0}\frac{e}{m_e\omega}\Big[iE_y - \frac{\omega_c}{\omega}E_x\Big]\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)^{-1}$$

$$(\omega^2 - c^2k^2)\Big(1 - \frac{\omega_c^2}{\omega^2}\Big)E_y = -i\omega_p^2\Big[iE_y - \frac{\omega_c}{\omega}E_x\Big] = \omega_p^2\ E_y + \frac{i\omega_p^2\omega_c}{\omega}E_x$$

$$-\frac{i\omega_p^2\omega_c}{\omega}E_x + \Big[(\omega^2 - c^2k^2)\Big(1 - \frac{\omega_c^2}{\omega^2}\Big) - \omega_p^2\Big]E_y = 0 \tag{6.141}$$

Defining a parameter

$$\alpha = \frac{\omega_p^2}{1 - \omega_c^2/\omega^2}$$

equations (6.140) and (6.141) can be written as

$$\Big(\omega^2 - c^2k^2 - \alpha\Big)E_x + \frac{i\alpha\omega_c}{\omega}E_y = 0$$

and

$$-\frac{i\alpha\omega_c}{\omega}E_x + \Big(\omega^2 - c^2k^2 - \alpha\Big)E_y = 0$$

For satisfying these simultaneous equations, we have

$$\left(\omega^2 - c^2k^2 - \alpha\right)^2 = -\frac{i\alpha\omega_c}{\omega}\frac{i\alpha\omega_c}{\omega} = (\alpha\omega_c/\omega)^2$$

$$\omega^2 - c^2k^2 - \alpha = \pm\alpha\omega_c/\omega$$

Thus,

$$\omega^2 - c^2k^2 = \alpha\left(1 \pm \frac{\omega_c}{\omega}\right) = \frac{\omega_p^2}{1-(\omega_c^2/\omega^2)}\left(1 \pm \frac{\omega_c}{\omega}\right)$$

$$= \omega_p^2\frac{1 \pm (\omega_c/\omega)}{[1+(\omega_c/\omega)][1-(\omega_c/\omega)]} = \frac{\omega_p^2}{1 \mp (\omega_c/\omega)}$$

The sign $\mp$ shows that there are two possible solutions corresponding to two different waves which propagate along $\vec{B}_0$. The dispersion relations are

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1-(\omega_c/\omega)} \qquad \text{R wave}$$

and

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1+(\omega_c/\omega)} \qquad \text{L wave}$$

The R and L waves are circularly polarized. R stands for right-hand circular polarization whereas L stands for left-hand circular polarization.

6.16 Hydromagnetic waves

In the preceding sections, we have accounted for high frequency electron waves. Now, we would like to discuss for low frequency ion waves. Out of several possible modes, we consider here two of them: (i) hydromagnetic waves along $\vec{B}_0$, called the Alfvén waves and (ii) magnetosonic waves across $\vec{B}_0$. In this section we shall discuss about the Alfvén waves.

In plane geometry, Alfvén wave has $\vec{k}$ along $\vec{B}_0$; $\vec{E}_1$ and $\vec{j}_1$ both perpendicular to $\vec{B}_0$; and both $\vec{B}_1$ and $\vec{u}_1$ perpendicular to both $\vec{B}_0$ and

$\vec{E}_1$ as shown in Figure 6.7.

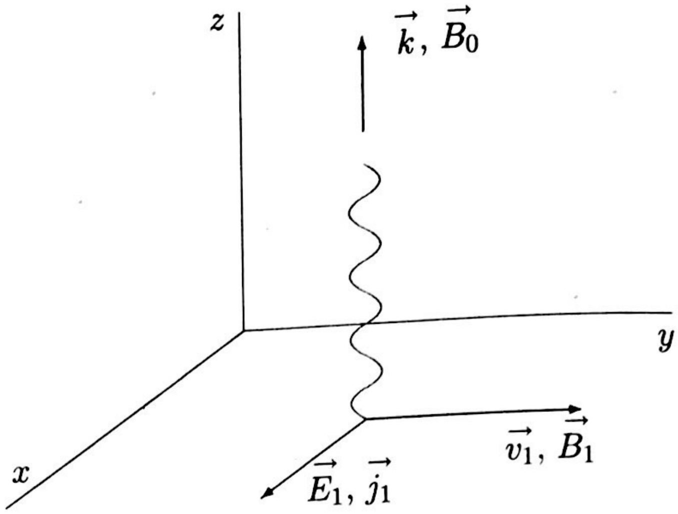

Figure 6.7: An Alfvén wave propagating along $\vec{B}_0$.

Since the magnetic field $\vec{B}_0$ is uniform and independent of time, we still have the Maxwell equations

$$\nabla\times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t} \qquad \text{and} \qquad c^2\nabla\times \vec{B}_1 = \frac{1}{\epsilon_0}\vec{j}_1 + \frac{\partial \vec{E}_1}{\partial t}$$

where $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light. As discussed earlier, these equations can be solved to get the equation

$$-\vec{k}\,(\vec{k}\cdot\vec{E}_1) + k^2\vec{E}_1 = \frac{i\omega}{c^2\epsilon_0}\vec{j}_1 + \frac{\omega^2}{c^2}\vec{E}_1 \tag{6.142}$$

Since $\vec{E}_1$ and $\vec{k}$ are perpendicular to each other, we have $\vec{k}\cdot\vec{E}_1 = 0$. Further, we have taken $\vec{E}_1$ along x-axis, and therefore only x-component of this equation is non-trivial. For low frequency waves, the current density $\vec{j}_1$ has contributions from both the electrons and ions as

$$\vec{j}_1 = n_0 e(\vec{u}_{i1} - \vec{u}_{e1}) \tag{6.143}$$

Using equation (6.143) in (6.142) and taking the x-component of the resultant equation, we have

$$\epsilon_0(\omega^2 - c^2k^2)E_1 = -i\omega n_0 e(u_{ix} - u_{ex}) \tag{6.144}$$

We assume also that there are no thermal motions ($T_i = 0$). The equations of motion is

$$m_i n_i\left[\frac{\partial \vec{u}_i}{\partial t} + (\vec{u}_i.\nabla)\vec{u}_i\right] = en_i(\vec{E} + \vec{u}_i \times \vec{B}_0)$$

On linearisation, this equation reduces to

$$m_i \frac{\partial \vec{u}_{i1}}{\partial t} = e\vec{E}_1 + e\vec{u}_{i1} \times \vec{B}_0 \tag{6.145}$$

We assume that the oscillating quantities behave sinusoidally as $e^{i(\vec{k}.\vec{r}-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by $i\vec{k}$. The x and y components of equation (6.145) are

$$-i\omega m_i u_{ix} = eE_1 + eu_{iy}B_0$$

and

$$-i\omega m_i u_{iy} = -eu_{ix}B_0$$

On solving these equations we get,

$$u_{ix} = \frac{ie}{m_i\omega}\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} E_1 \tag{6.146}$$

where $\Omega_c = eB_0/m_i$ is the ion cyclotron frequency. For the equation of motion for electrons, u_{ex} can be written from this expression by replacing m_i by m_e, Ω_c by ω_c, e by $-e$, and thus, we have

$$u_{ex} = -\frac{ie}{m_e\omega}\left(1 - \frac{\omega_c^2}{\omega^2}\right)^{-1} E_1$$

For the situation, $\omega^2 << \omega_c^2$, we have

$$u_{ex} = \frac{ie}{m_e\omega}\frac{\omega^2}{\omega_c^2}E_1 \rightarrow 0 \tag{6.147}$$

For $\omega_c^2 >> \omega^2$, Larmor gyrations of electrons are neglected and the electrons have $\vec{E} \times \vec{B}$ drift in the y-direction. Using equations (6.146) and (6.147) in (6.144), we get

$$\epsilon_0(\omega^2 - c^2k^2)E_1 = -i\omega n_0 e\frac{ie}{m_i\omega}\left(1 - \frac{\Omega_c^2}{\omega^2}\right)^{-1} E_1 \tag{6.148}$$

Since $E_1 \neq 0$, and defining the ion plasma frequency $\Omega_p^2 = ne^2/\epsilon_0 m_i$, we get

$$\omega^2 - c^2k^2 = \Omega_p^2\Big(1 - \frac{\Omega_c^2}{\omega^2}\Big)^{-1}$$

Further, we assume that hydromagnetic waves have frequencies well below the ion cyclotron frequency ($\omega^2 << \Omega_c^2$), so that

$$\omega^2 - c^2k^2 = \Omega_p^2\Big(-\frac{\omega^2}{\Omega_c^2}\Big) = -\omega^2\frac{\Omega_p^2}{\Omega_c^2}$$

$$= -\omega^2\frac{n_0e^2}{\epsilon_0 m_i}\frac{m_i^2}{e^2B_0^2} = -\omega^2\frac{\rho}{\epsilon_0 B_0^2}$$

Hence,

$$\omega^2\Big[1 + \frac{\rho}{\epsilon_0 B_0^2}\Big] = c^2k^2$$

$$\frac{\omega^2}{k^2} = \frac{c^2}{1 + (\rho/\epsilon_0 B_0^2)} = \frac{c^2}{1 + (\rho\mu_0/B_0^2)c^2} \tag{6.149}$$

where $\rho = n_0 m_i$ is the mass density. The denominator on the right side of equation (6.149) may be interpreted as relative dielectric constant for low frequency perpendicular motions. Equation (6.149) may be approximated as

$$\frac{\omega^2}{k^2} = \frac{c^2}{(\rho\mu_0/B_0^2)c^2} = \frac{B_0^2}{\rho\mu_0} \qquad \frac{\omega}{k} = v_\phi = \frac{B_0}{(\rho\mu_0)^{1/2}}$$

These hydromagnetic waves travel along $\vec{B_0}$ at a constant velocity v_A, called the Alfvén velocity

$$v_A \equiv \frac{B_0}{(\rho\mu_0)^{1/2}}$$

6.17 Magnetosonic waves

Here, we consider low frequency electromagnetic waves propagating across the magnetic field $\vec{B_0}$ (Figure 6.8). We consider $\vec{B_0} = B_0\hat{k}$ and $\vec{E_1} = E_1\hat{i}$, but now let $\vec{k} = k\hat{j}$ so that the $\vec{E_1} \times \vec{B_0}$ drifts lie along $\vec{k}$. Hence, the plasma will be compressed and released along y-axis during the course of oscillations. Therefore, we have to include the pressure term ∇p along

the y-axis in the equation of motion. Since the magnetic field $\vec{B_0}$ is uniform and independent of time, we still have the Maxwell equations

$$\nabla\times \vec{E_1} = -\frac{\partial \vec{B_1}}{\partial t} \qquad \text{and} \qquad c^2\nabla\times \vec{B_1} = \frac{1}{\epsilon_0}\vec{j_1} + \frac{\partial \vec{E_1}}{\partial t}$$

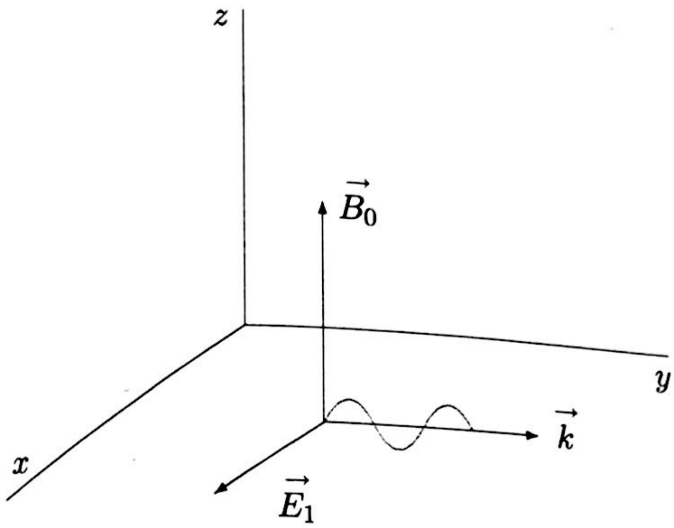

Figure 6.8: A magnetosonic wave propagating at right angle to $\vec{B_0}$.

where $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light. As discussed earlier, these equations can be solved to get the equation

$$-\vec{k}\,(\vec{k}.\vec{E_1}) + k^2\vec{E_1} = \frac{i\omega}{c^2\epsilon_0}\vec{j_1} + \frac{\omega^2}{c^2}\vec{E_1} \tag{6.150}$$

Since $\vec{E_1}$ and $\vec{k}$ are perpendicular to each other, we have $\vec{k}\,.\,\vec{E_1} = 0$. Further, we have taken $\vec{E_1}$ along x-axis, and therefore only x-component of this equation is non-trivial. For low frequency waves, the current density $\vec{j_1}$ has contributions from both the electrons and ions as

$$\vec{j_1} = n_0e(\vec{v_{i1}} - \vec{v_{e1}}) \tag{6.151}$$

Using equation (6.151) in (6.150) and taking the x-component of the resultant equation, we have

$$\epsilon_0(\omega^2 - c^2k^2)E_1 = -i\omega n_0e(u_{ix} - u_{ex}) \tag{6.152}$$

The equations of motion and continuity are

$$m_i n_i\left[\frac{\partial \vec{u_i}}{\partial t} + (\vec{u_i}.\nabla)\vec{u_i}\right] = en_i(\vec{E} + \vec{u_i} \times \vec{B_0}) - \gamma_i KT_i \nabla n_i$$

and

$$\frac{\partial n_i}{\partial t} + \nabla.(n_i \vec{u_i}) = 0$$

On linearisation, these equations reduce to

$$m_i n_0 \frac{\partial \vec{u_{i1}}}{\partial t} = en_0(\vec{E_1} + \vec{u}_{i1} \times \vec{B_0}) - \gamma_i KT_i \nabla n_{i1} \tag{6.153}$$

and

$$\frac{\partial n_{i1}}{\partial t} + n_0 \nabla.\vec{u_{i1}} = 0 \tag{6.154}$$

We assume that the oscillating quantities behave sinusoidally as $e^{i(\vec{k}.\vec{r}-\omega t)}$. Then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by $i\vec{k}$. Expressing $\vec{u_{i1}} = u_{ix}\hat{i} + u_{iy}\hat{j} + u_{iz}\hat{k}$, the x and y components of equation (6.153) are

$$-i\omega m_i n_0 u_{ix} = en_0(E_1 + u_{iy}B_0) \tag{6.155}$$

and

$$-i\omega m_i n_0 u_{iy} = -en_0 u_{ix} B_0 - \gamma_i KT_i ikn_{i1}$$

Remember that the density variation is along the y-axis. Thus, we have

$$u_{ix} = \frac{ie}{m_i\omega}(E_1 + u_{iy}B_0) \tag{6.156}$$

and

$$u_{iy} = \frac{ie}{m_i\omega}(-u_{ix}B_0) + \frac{k}{\omega}\frac{\gamma_i KT_i}{m_i}\frac{n_{i1}}{n_0} \tag{6.157}$$

Linearisation of equation (6.154) gives

$$-i\omega n_{i1} = -n_0 iku_{iy} \qquad \frac{n_{i1}}{n_0} = \frac{k}{\omega}u_{iy} \tag{6.158}$$

Using equation (6.158) in (6.157), we get

$$v_{iy} = -\frac{ie}{m_i\omega}v_{ix}B_0 + \frac{k^2}{\omega^2}\frac{\gamma_i KT_i}{m_i}u_{iy} = -\frac{i\Omega_c}{\omega}v_{ix} + \alpha_i v_{iy}$$

$$v_{iy}(1-\alpha_i) = -\frac{i\Omega_c}{\omega}u_{ix} \qquad v_{iy} = -\frac{i\Omega_c}{\omega}u_{ix}(1-\alpha_i)^{-1} \tag{6.159}$$

where

$$\Omega_c = \frac{eB_0}{m_i} \qquad \text{and} \qquad \alpha_i = \frac{k^2}{\omega^2}\frac{\gamma_i KT_i}{m_i}$$

Using equation (6.159) in (6.156), we get

$$v_{ix} = \frac{ie}{m_i\omega}E_1 + \frac{i\Omega_c}{\omega}\Big(-\frac{i\Omega_c}{\omega}\Big)(1-\alpha_i)^{-1}u_{ix} = \frac{ie}{m_i\omega}E_1 + \frac{\Omega_c^2}{\omega^2}(1-\alpha_i)^{-1}v_{ix}$$

$$v_{ix}\Big(1 - \frac{\Omega_c^2/\omega^2}{1-\alpha_i}\Big) = \frac{ie}{m_i\omega}E_1$$

$$v_{ix}\Big(\frac{1-\alpha_i-\Omega_c^2/\omega^2}{1-\alpha_i}\Big) = \frac{ie}{m_i\omega}E_1 \tag{6.160}$$

Equation for electron, corresponding to equation (6.160) can be obtained by replacing e by $-e$, m_i by m_e, Ω_c by ω_c and α_i by α_e as

$$u_{ex}\Big(\frac{1-\alpha_e-\omega_c^2/\omega^2}{1-\alpha_e}\Big) = -\frac{ie}{m_e\omega}E_1 \tag{6.161}$$

where

$$\omega_c = \frac{eB_0}{m_e} \qquad \text{and} \qquad \alpha_e = \frac{k^2}{\omega^2}\frac{\gamma_e KT_e}{m_e}$$

for $\omega^2 << \omega_c^2$ and $\omega^2 << k^2\ u_{th}^2$, we have

$$u_{ex}\Big(\frac{-\omega_c^2/\omega^2}{1-\alpha_e}\Big) = -\frac{ie}{m_e\omega}E_1 \qquad u_{ex} = \frac{ie}{m_e\omega}\frac{\omega^2}{\omega_c^2}(1-\alpha_e)E_1$$

$$u_{ex} = \frac{ie}{m_e\omega}\frac{\omega^2}{\omega_c^2}\Big(1-\frac{k^2}{\omega^2}\frac{\gamma_e KT_e}{m_e}\Big)E_1 \rightarrow -\frac{ik^2}{\omega B_0^2}\frac{\gamma_e KT_e}{e}E_1 \tag{6.162}$$

Using equations (6.161) and (6.162) in (6.152), we have

$$\epsilon_0(\omega^2 - c^2k^2)E_1 = -i\omega n_0 e\Big[\frac{ie}{m_i\omega}\Big(\frac{1-\alpha_i}{1-\alpha_i-\Omega_c^2/\omega^2}\Big) + \frac{ik^2}{\omega B_0^2}\frac{\gamma_e KT_e}{e}\Big]E_1$$

For $\omega^2 << \Omega_c^2$, we can neglect $1-\alpha_i$ in comparison to Ω_c^2/ω^2 and for $E_1 \neq 0$, we have

$$\omega^2 - c^2k^2 = -\frac{i\omega n_0 e}{\epsilon_0}\Big[\frac{ie}{m_i\omega}\Big(\frac{1-\alpha_i}{-\Omega_c^2/\omega^2}\Big) + \frac{ik^2\gamma_e KT_e}{\omega B_0^2 e}\Big]$$

$$= -\frac{n_0e^2}{m_i\epsilon_0}\frac{\omega^2}{\Omega_c^2}(1-\alpha_i) + \frac{n_0k^2m_i\gamma_e KT_e}{\epsilon_0 B_0^2 m_i}$$

$$= -\frac{\Omega_p^2\omega^2}{\Omega_c^2}(1-\alpha_i) + \frac{c^2\mu_0\rho k^2\gamma_e KT_e}{B_0^2 m_i}$$

$$= -\frac{\Omega_p^2\omega^2}{\Omega_c^2}(1-\alpha_i) + \frac{c^2k^2\gamma_e KT_e}{v_A^2 m_i}$$

Thus, we have

$$\omega^2 - c^2k^2 - \frac{c^2k^2}{v_A^2}\frac{\gamma_e KT_e}{m_i} + \frac{\Omega_p^2}{\Omega_c^2}\left(\omega^2 - \omega^2\alpha_i\right) = 0$$

$$\omega^2 - c^2k^2\left(1 + \frac{\gamma_e KT_e}{m_i v_A^2}\right) + \frac{\Omega_p^2}{\Omega_c^2}\left(\omega^2 - k^2\frac{\gamma_i KT_i}{m_i}\right) = 0$$

Now, $$\frac{\Omega_p^2}{\Omega_c^2} = \frac{n_0 e^2}{m_i\epsilon_0}\frac{m_i^2}{e^2B_0^2} = \frac{\mu_0 c^2\rho}{B_0^2} = \frac{c^2}{v_A^2}$$

Thus, we have

$$\omega^2\left(1 + \frac{c^2}{v_A^2}\right) = c^2k^2\left(1 + \frac{\gamma_e KT_e}{m_i v_A^2} + \frac{\gamma_i KT_i}{m_i v_A^2}\right)$$

$$= c^2k^2\left(1 + \frac{\gamma_e KT_e + \gamma_i KT_i}{m_i v_A^2}\right) = c^2k^2\left(1 + \frac{v_s^2}{v_A^2}\right)$$

Therefore,

$$\omega^2\left(\frac{v_A^2 + c^2}{v_A^2}\right) = c^2k^2\left(\frac{v_A^2 + v_s^2}{v_A^2}\right) \qquad \frac{\omega^2}{k^2} = c^2\frac{v_s^2 + v_A^2}{c^2 + v_A^2}$$

This is the dispersion relation for magnetosonic waves propagating perpendicular to $\vec{B}_0$. In this acoustic wave, compressions and rarefactions are produced not by the motions along $\vec{E}$, but perpendicular to $\vec{E}$ by the $\vec{E} \times \vec{B}_0$ drift. When $\vec{B}_0 \to 0$, $v_A \to 0$, and we have $(\omega/k) = v_s$. Hence, the magnetosonic wave turns into an ordinary ion acoustic wave. When $KT_i \to 0$ and $K\,T_e \to 0$, $v_s \to 0$, and we have

$$\frac{\omega^2}{k^2} = \frac{c^2v_A^2}{c^2 + v_A^2} \qquad \frac{\omega}{k} = \frac{v_A}{\sqrt{1 + v_A^2/c^2}}$$

Thus, the magnetosonic wave becomes a modified Alfvén wave, whose phase velocity is smaller than the Alfvén velocity. When v_s is finite, phase velocity of the wave is

$$\frac{\omega}{k} = \sqrt{\frac{v_s^2 + v_A^2}{1 + v_A^2/c^2}}$$

is generally larger than the Alfvén velocity, and therefore, the wave is often called the fast hydromagnetic wave.

6.18 Problems and questions

1. Derive an expression for plasma frequency. What happens when thermal motions are accounted for.

2. Derive an expression for dispersion relation for electron plasma waves. Show that the group velocity is never greater than the speed of light, and at large k the information travels faster than that at the small k.

3. Derive an expression for the velocity of sound waves in an ordinary gas.

4. Show that in the electron plasma waves the phase velocity is always greater than or equal to $\sqrt{3/2}\ v_{th}$ whereas the group velocity is always less than or equal to $\sqrt{3/2}\ v_{th}$, where v_{th} is thermal velocity of electrons.

5. Derive an expression for the velocity of ion waves in a plasma. Estimate the error introduced by the plasma approximation.

6. Derive the dispersion relation for ordinary and extra-ordinary electromagnetic waves perpendicular to $\vec{B}_0$

7. Derive expression for the velocity of Alfvén waves.

8. Derive expression for the velocity of the Magnetosonic waves.

9. Write short notes on the following

    (i) Phase velocity

    (ii) Group velocity

    (iii) Plasma frequency

7

Diffusion and Resistivity

In the preceding chapters, we have considered infinite homogeneous plasma in equilibrium. However, a realistic plasma is not homogeneous, but there is density gradient leading to diffusion of plasma from a high density region towards a low density region. Main problem in the controlled thermonuclear reactions is to reduce the rate of diffusion by using magnetic field. Here, we shall first discuss the process of diffusion in absence of magnetic field. And then we shall include the magnetic field. As a simplification, we assume that the plasma is weakly ionized and thus there is a large number of neutral particles. Hence, the charged particles collide primarily with neutral particles rather than with one another. The case of fully ionized plasma would be considered in the later part of this chapter. Thus, we have a non-uniform distribution of positive ions and electrons in a dense background of neutral particles.

When the plasma spreads out as a result of pressure-gradient (owing to the density-gradient) and electric field forces, individual particles undergo a random walk and to collide frequently with the neutral particles (Figure 7.1). Let us first have knowledge of collision and diffusion parameters.

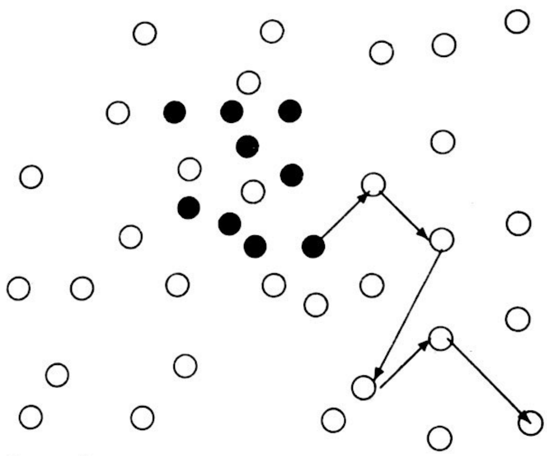

Figure 7.1: Diffusion of gas particles through random collisions. Hollow circles represent neutral particles whereas solid circles represent charged particles.

7.1 Parameters

Here, we shall discuss about some parameters in the process of diffusion.

7.1.1 Collision parameters

When an electron collides with a neutral particle, it may lose a fraction of its momentum depending upon the angle at which it is deviated. In a head-on collision with a heavy particle, its direction of motion is reversed and thus it loses twice of its initial momentum. The probability of momentum lose can be expressed in terms of a parameter, called the cross section σ that the particles would have if they were perfect absorber of momentum. Consider a slab of area A and thickness dx containing n_n neutral particles per m^3. Suppose the electrons incident upon the slab and lose the momentum after colliding with the particles. Let us consider particles to be opaque spheres of cross sectional area σ. That is when an electron comes within the (considered) area blocked by the particle, the electron loses all of its momentum. The number of particles in the slab (of volume $A\, dx$) is $n_n A\, dx$.

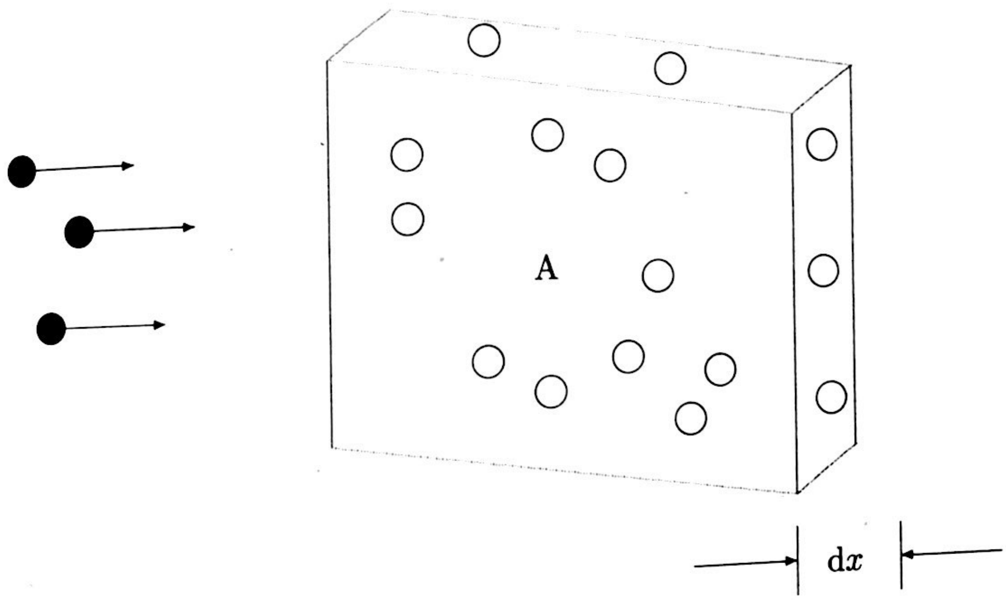

Figure 7.2: Shows a cross section.

The fractional area blocked by the particles is

$$\frac{\sigma(n_n A\, dx)}{A} = \sigma n_n\, dx$$

Here, we are assuming that the cross sectional areas of the particles do not overlap. Suppose $\vec{\Phi}$ is the flux of electrons incident on the surface of area A of the slab, the flux absorbed by the slab is

$$\vec{\Phi}\sigma n_n \,\mathrm{d}x$$

Thus, the flux emerging on the other side of the slab is

$$\vec{\Phi}' = \vec{\Phi} - \vec{\Phi}\sigma n_n \,\mathrm{d}x$$

Hence, the change of flux per unit thickness of the slab is

$$\frac{\vec{\Phi}' - \vec{\Phi}}{\mathrm{d}x} = -\sigma n_n \vec{\Phi} \qquad \frac{\mathrm{d}\vec{\Phi}}{\mathrm{d}x} = -\sigma n_n \vec{\Phi}$$

The negative sign indicates that the flux decreases with the increase of x. On integration, we get

$$\vec{\Phi} = \vec{\Phi}_0 \,\mathrm{e}^{-\sigma n_n x} = \vec{\Phi}_0 \,\mathrm{e}^{-x/\lambda_m}$$

where $\vec{\Phi}_0$ is the flux at $x = 0$ and λ_m is the distance at which the flux reduces to $1/e$ of its initial value. The parameter λ_m is defined as the *mean free path* for collisions

$$\lambda_m = \frac{1}{\sigma n_n}$$

Hence, after traveling a distance λ_m the electron has a good probability for colliding with an particle. For an electron moving with velocity v, the mean time between two successive collisions is

$$\tau = \frac{\lambda_m}{v}$$

and thus the mean frequency of collisions is

$$\nu = \tau^{-1} = \frac{v}{\lambda_m} = n_n \sigma v$$

When the velocity of electrons is averaged over the Maxwellian distribution, the collision frequency ν is

$$\nu = n_n \sigma \,\bar{v}$$

7.1.2 Diffusion parameters

Equation of motion of a particle, including collisions is

$$mn\frac{d\vec{v}}{dt} = mn\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}\right] = \pm en\vec{E} - \nabla p - mn\nu\vec{v} \tag{7.1}$$

Here, $\pm$ indicates the sign of the charge of the particle. In order to make equation (7.1) to be useful, the quantity ν must however be assumed to be constant. We consider a steady state in which $\partial\vec{v}/\partial t$ is zero. If the velocity $\vec{v}$ is sufficiently small (or ν is sufficiently large), a fluid element will not move into regions of different $\vec{E}$ or ∇p in a collision time, and therefore, $d\vec{v}/dt$ will also vanish. Putting the left side of equation (7.1) zero, we have (using $p = nKT$)

$$0 = \pm en\vec{E} - KT\nabla n - mn\nu\vec{v}$$

$$\vec{v} = \frac{1}{mn\nu}\left(\pm en\vec{E} - KT\nabla n\right)$$

$$= \pm\frac{e}{m\nu}\vec{E} - \frac{KT}{m\nu}\frac{\nabla n}{n} = \pm\mu\vec{E} - D\frac{\nabla n}{n} \tag{7.2}$$

where μ and D are mobility and diffusion coefficients, respectively, defined as

$$\mu = \frac{|q|}{m\nu} \qquad \text{and} \qquad D = \frac{KT}{m\nu}$$

These coefficients depend on the particle and are related through the *Einstein relation*

$$\mu = \frac{|q|}{KT}D$$

Using these coefficients, the flux $\vec{\Phi}_j$ of the j-th specie can be written

$$\vec{\Phi}_j = n_j v_j = \pm\mu_j n_j\vec{E} - D_j\nabla n_j \tag{7.3}$$

Here, we have used equation (7.2). When either the electric field $\vec{E} = 0$ or the specie is uncharged so that $\mu = 0$, we have

$$\vec{\Phi} = -D\nabla n$$

This is *Fick's law* and expresses that in the diffusion process a net flux moves from a more dense region to a less dense region and the flux is proportional to the density-gradient.

7.2 Decay of plasma by diffusion

7.2.1 Ambipolar diffusion

When some particles of a plasma produced in a container diffuse to the walls of the container, the oppositely charged particles (ions and electrons) combine after reaching there. Hence, the density of the charged particles near the wall is essentially zero. In the preceding chapters we have used fluid equations of motion and continuity to study the behaviour of a plasma. If the decay is slow, we need only to keep the time derivative in the continuity equation. Thus, we have

$$\frac{\partial n_j}{\partial t} + \nabla . \vec{\Phi}_j = 0 \tag{7.4}$$

where $\vec{\Phi}_j$ is given by the equation (7.3). It is obvious that the flux of ions $\vec{\Phi}_i$ must be equal to that of the electrons $\vec{\Phi}_e$, otherwise there would be a serious charge imbalance. For a plasma of size larger than the Debye length, it must be a quasi-neutral and the rate of diffusion of ions and electrons would somehow adjust themselves so that the ions and electrons leave at the same rate. This phenomenon can be visualized in an easy manner as the following. The electrons being lighter have higher thermal velocities and intend to leave the plasma first. And hence, the ions are left behind. Through the separation of ions and electrons, an electric field develops in such a way that it retards the rate of the loss of electrons and accelerates the rate of the loss of ions. This electric field $\vec{E}$ can be obtained by putting $\vec{\Phi}_i = \vec{\Phi}_e = \vec{\Phi}$

$$\vec{\Phi} = \mu_i n \vec{E} - D_i \nabla n = -\mu_e n \vec{E} - D_e \nabla n \tag{7.5}$$

$$n \vec{E} (\mu_i + \mu_e) = (D_i - D_e) \nabla n$$

$$\vec{E} = \frac{D_i - D_e}{\mu_i + \mu_e} \frac{\nabla n}{n} \tag{7.6}$$

Here, we have used $n_i = n_e = n$. Using equation (7.6) in (7.5), we have

$$\vec{\Phi} = \mu_i n \frac{D_i - D_e}{\mu_i + \mu_e} \frac{\nabla n}{n} - D_i \nabla n$$

$$= \frac{\mu_i D_i - \mu_i D_e - \mu_i D_i - \mu_e D_i}{\mu_i + \mu_e} \nabla n$$

$$= -\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e} \nabla n = -D_a \nabla n \tag{7.7}$$

This is Fick's law with a new diffusion coefficient

$$D_a = \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e}$$

known as the *ambipolar diffusion coefficient.* If D_a is constant, using equation (7.7) in (7.4), we have

$$\frac{\partial n}{\partial t} = D_a \nabla^2 n \tag{7.8}$$

The magnitude of D_a can be estimated in the following manner. Since the mass of an electron is much smaller than that of an ion, we have $\mu_e >> \mu_i$ and therefore

$$D_a \approx \frac{\mu_i D_e + \mu_e D_i}{\mu_e} = D_i + \frac{\mu_i}{\mu_e} D_e \approx D_i$$

Using the Einstein relation, we have

$$D_a = D_i + \frac{D_i}{KT_i} \frac{KT_e}{D_e} D_e = D_i + \frac{T_e}{T_i} D_i$$

On thermalization, we have $T_i = T_e$, and $D_a = 2D_i$. Thus, the effect of ambipolar electric field is to enhance the diffusion of ions by a factor of two. However, the diffusion rate of the two species together is primarily controlled by the slower species ($D_a \approx D_i$).

Solution of equation (7.8)

Equation (7.8) can be solved by the method of the separation of variables

$$n(\vec{r}, t) = T(t)\ S(\vec{r}) \tag{7.9}$$

Using equation (7.9) in (7.8) (for convenience, the suffix a is dropped from D), we get

$$S \frac{\partial T}{\partial t} = DT \nabla^2 S$$

$$\frac{1}{T} \frac{\partial T}{\partial t} = \frac{D}{S} \nabla^2 S \tag{7.10}$$

In equation (7.10), the left side depends on t whereas the right side depends on $\vec{r}$. Hence, the equation can only be satisfied when each side is equal to some constant, say $-1/\tau$. Thus, for the left side of equation (7.10), we have

$$\frac{1}{T}\frac{dT}{dt} = -\frac{1}{\tau} \qquad \frac{dT}{T} = -\frac{1}{\tau}\,dt$$

so that

$$T = T_0\, e^{-t/\tau} \tag{7.11}$$

where $T = T_0$ at $t = 0$. It shows that density of the charged particles decreases exponentially with time. For the right side of equation (7.10), we have

$$\frac{D}{S}\,\nabla^2 S = -\frac{1}{\tau} \qquad \nabla^2 S = -\frac{1}{D\tau}S \tag{7.12}$$

Equation (7.12) can be solved for various geometries of the container of plasma.

Diffusion in a slab

In a slab, for variation in one-dimension, equation (7.12) can be written as

$$\frac{d^2S}{dx^2} = -\frac{1}{D\tau}S$$

General solution of this equation is

$$S = A\cos\left(\frac{x}{\sqrt{D\,\tau}}\right) + B\sin\left(\frac{x}{\sqrt{D\tau}}\right)$$

where A and B are constants. Since we expect solution to be symmetrical about the $x = 0$ plane, we can ignore the sine term and thus we have

$$S = A\cos\left(\frac{x}{\sqrt{D\tau}}\right)$$

Since, we expect the density of the charged particles to be nearly zero at the walls $x = \pm l$ of the container, we have

$$0 = A\cos\left(\frac{l}{\sqrt{D\tau}}\right)$$

Since we are interested in non-zero value of A, we have

$$\frac{l}{\sqrt{D\tau}} = (2n+1)\ \frac{\pi}{2}$$

where n is an integer. We account for the simplest case of $n = 0$, and hence,

$$S = A\cos\left(\frac{x\pi}{2l}\right) \tag{7.13}$$

Using equations (7.11) and (7.13) in (7.9), we get

$$n = T_0\ \mathrm{e}^{-t/\tau} A\cos\left(\frac{x\pi}{2l}\right) = n_0\ \mathrm{e}^{-t/\tau}\cos\left(\frac{x\pi}{2l}\right)$$

where $n_0 (= T_0 A)$ is the number of charged particles at $t = 0$ and $x = 0$.

Diffusion in a cylinder

For a cylinder, equation (7.12) can be written as

$$\frac{\mathrm{d}^2S}{\mathrm{d}r^2} + \frac{1}{r}\ \frac{\mathrm{d}S}{\mathrm{d}r} + \frac{1}{D\tau}S = 0$$

$$r^2\frac{\mathrm{d}^2S}{\mathrm{d}r^2} + r\frac{\mathrm{d}S}{\mathrm{d}r} + \frac{r^2}{D\tau}\ S = 0 \tag{7.14}$$

Here, we are considering radial variation of the density of the charged particles. That is we are considering no variation with respect to θ and z. Let $r = x\sqrt{D\tau}$. Then, we have

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{1}{\sqrt{D\tau}}$$

$$\frac{\mathrm{d}S}{\mathrm{d}r} = \frac{\mathrm{d}S}{\mathrm{d}x}\ \frac{\mathrm{d}x}{\mathrm{d}r} = \frac{1}{\sqrt{D\tau}}\frac{\mathrm{d}S}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2S}{\mathrm{d}r^2} = \frac{1}{\sqrt{D\tau}}\ \frac{\mathrm{d}^2S}{\mathrm{d}x^2}\ \frac{\mathrm{d}x}{\mathrm{d}r} = \frac{1}{D\tau}\frac{\mathrm{d}^2S}{\mathrm{d}x^2}$$

Using these relations, equation (7.14) can be written as

$$x^2\frac{\mathrm{d}^2S}{\mathrm{d}x^2} + x\frac{\mathrm{d}S}{\mathrm{d}x} + x^2S = 0$$

This is the zeroth order Bessel equation and its solution is the Bessel function $J_0(x)$. Thus the solution of equation (7.14) is

$$S = AJ_0(x) = AJ_0\left(\frac{r}{\sqrt{D\tau}}\right) \tag{7.15}$$

where A is a constant. Since, we expect the density of the charged particles to be nearly zero ($S = 0$) at the surface of the cylinder $r = R$, we have

$$J_0\left(\frac{R}{\sqrt{D\tau}}\right) = 0$$

For the first zero of the Bessel function, we have

$$\frac{R}{\sqrt{D\tau}} = 2.4 \tag{7.16}$$

Using equations (7.16) in (7.15), we get

$$S = AJ_0\left(\frac{2.4r}{R}\right) \tag{7.17}$$

Using equations (7.11) and (7.17) in (7.9), we get

$$n = T_0\ \mathrm{e}^{-t/\tau} AJ_0\left(\frac{2.4r}{R}\right) = n_0\ \mathrm{e}^{-t/\tau} J_0\left(\frac{2.4r}{R}\right)$$

where $n_0(= T_0 A)$ is the density of the charged particles at $t = 0$ and $r = 0$.

7.2.2 Steady state solutions

In many experiments, it is possible to compensate the losses and thereby to maintain plasma in a steady state through continuous ionization or through injection of plasma. For calculating density profile in this case, we add a source term to the equation of continuity:

$$\frac{\partial n}{\partial r} - D\nabla^2 n = Q(\vec{r})$$

Here, positive sign of the source function $Q(\vec{r})$ represents a source that contributes to positive $\partial n/\partial r$. We shall consider two cases for the source: (i) Plane source and (ii) Line source.

(i) Plane source

Let us consider a localized source on the plane $x = 0$ and then calculate the profile. Such a source may be, for example, a slit-collimated beam of ultraviolet light which is strong enough to ionize the neutral gas. Now, the steady state diffusion equation is

$$\frac{\mathrm{d}^2 n}{\mathrm{d}x^2} = -\frac{Q}{D}\,\delta(0)$$

Thus, except at $x = 0$, the density satisfies the equation

$$\frac{d^2 n}{dx^2} = 0$$

Integration of this equation twice, we get

$$n = Cx + D$$

On applying the conditions: (i) At $x = 0$, we have $n = n_0$ and (ii) At $|x| = L$, we have $n = 0$, we get

$$n = n_0\left(1 - \frac{|x|}{L}\right)$$

It shows that for a plane source, the plasma has a linear profile as shown in Figure 7.3.

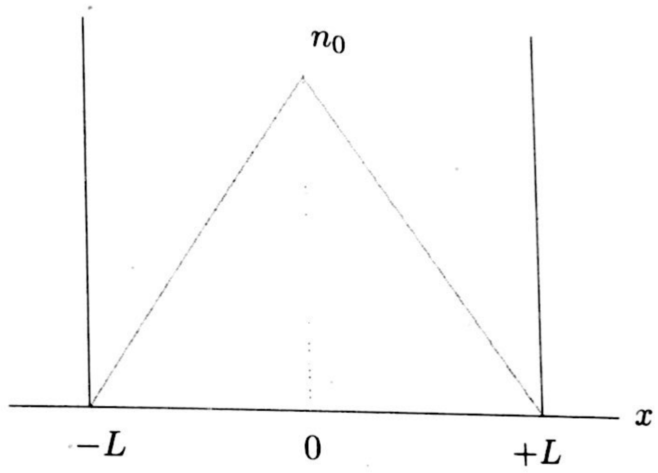

Figure 7.3: Density profile resulting from a plane source under diffusion.

(ii) Line source

Let us consider a localized source along a line. For this kind of source, the plasma is cylindrical and the source is along the axis of the cylinder. Such a source may be, for example, a beam of energetic electrons producing ionization along the axis. Let us calculate the profile of plasma. Now, the steady state diffusion equation, for cylindrical symmetry, is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial n}{\partial r}\right) = -\frac{Q}{D}\,\delta(0)$$

Thus, except at $r = 0$, the density satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial n}{\partial r}\right) = 0$$

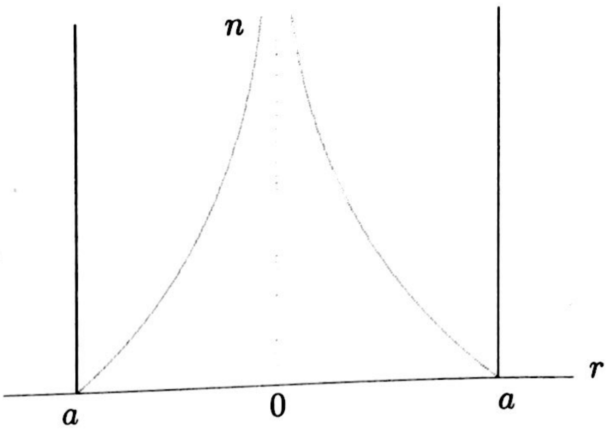

Figure 7.4: Density profile resulting from a line source under diffusion.

Integration of this equation gives

$$\frac{\partial n}{\partial r} = \frac{a}{r}$$

where a is constant. Integration of this equation gives

$$n = \ln(a/r)$$

where we applied that condition that density $n = 0$ at $r = a$. The profile of density in this case is shown in Figure 7.4.

7.3 Recombination

In a plasma, when an ion and an electron collide, particularly at low relative velocity, there is a finite probability of their recombination into a neutral particle. The amount of energy released in the process is either emitted in the form of a photon (the process is called the *radiative recombination*) or given to a third body participating in the process (the process is called the *three-body recombination*). Owing to recombination, density of charged particles in the plasma decreases. This rate of decrease is obviously proportional to the density of the ions and that of the electrons, *i.e.*, proportional to $n_i n_e = n^2$. Thus, we have

$$-\frac{\partial n}{\partial t} = \alpha n^2 \qquad\qquad -\frac{\mathrm{d}n}{n^2} = \alpha\ \mathrm{d}t$$

where α is known as the *recombination coefficient*. Integration of this equation gives

$$\frac{1}{n} = \frac{1}{n_0} + \alpha t$$

where n_0 is the density of the charged particles at $t = 0$. When $n << n_0$, *i.e.*, the density has fallen much below the initial density, we have

$$\frac{1}{n} = \alpha t \qquad\qquad n \propto \frac{1}{\alpha t}$$

It shows that in the recombination process, the density of charged particles is inversely proportional to the time.

Remember that for the diffusion process, the decay of the density of charged particles was exponential showing that the diffusion is more effective than the recombination.

7.4 Diffusion across a magnetic field

We have yet discussed about the diffusion in absence of an applied magnetic field. Let us now consider a weakly ionized plasma in a magnetic field $\vec{B}_0$.

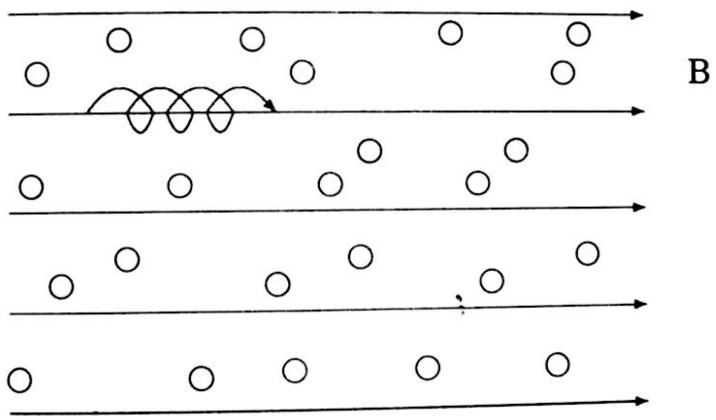

Figure 7.5: A charged particle in magnetic field gyrates about the same line of force until it collides with some particle.

Since the magnetic field does not affect the motion of charged particles moving along the field lines. The flux along the field lines of the j-th specie is

$$\Phi_{jz} = \pm\mu_j n_j E_z - D_j \frac{\partial n_j}{\partial z}$$

where μ_j and D_j are mobility and diffusion coefficients, respectively. In absence of collisions, the particles would not diffuse at all in the direction perpendicular to the field lines. There are of course the particle drifts across $\vec{B}_0$ due to electric field and due to gradient in the magnetic field. However, the motions can be arranged parallel to the walls. For example, in a symmetric cylinder (Figure 7.6), the gradients are all in the radial direction so that the drifts of the guiding centers are in the azimuthal direction. Since the drifts are not moving the guiding centers towards the surface of the cylinder, hence they are harmless.

When a charged particle collides with a neutral particle, it walks in a random manner, and some of the particles can move across the magnetic field towards the walls. The particle continues to gyrate about the magnetic field in the same direction, but the position of its guiding center shifts. (Larmor radius may also change, but we assume that the particle does not gain or loss energy on the average.) The particles thus diffuse in the direction opposite to ∇n. The step length in the random walk is no longer as it was in the magnetic-free situation, but has instead the magnitude of the Larmor radius $r_L(= mv_\perp/eB)$. Hence, the diffusion across $\vec{B}_0$ can be slowed down by increasing the magnetic field strength.

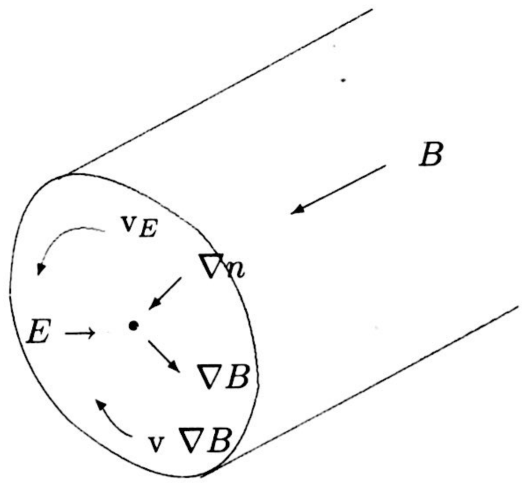

Figure 7.6: Particle drifts in a cylindrically symmetric plasma column do not give losses.

To see how this can be possible, let us write down perpendicular component of the fluid equation of motion for either species as the fol-

lowing

$$mn\,\frac{d\vec{v_\perp}}{dt} = \pm en(\vec{E} + \vec{v_\perp} \times \vec{B_0}) - KT\nabla n - mn\nu\vec{v_\perp} \tag{7.18}$$

Here, $\pm$ indicates the sign of the charge of the particle. In order to make equation (7.18) to be useful, the quantity ν must however be assumed to be constant. We consider the steady state in which $\partial\vec{v}/\partial t$ is zero. If the velocity $\vec{v}$ is sufficiently small (or ν is sufficiently large), a fluid element will not move into regions of different $\vec{E}$ or ∇p in a collision time, and therefore, $\mathrm{d}\,\vec{v}/\mathrm{d}t$ will also vanish. Putting the left side of equation (7.18) zero, we have

$$0 = \pm en(\vec{E} + \vec{v_\perp} \times \vec{B_0}) - KT\nabla n - mn\nu\vec{v_\perp}$$

$$mn\nu\,\vec{v_\perp} = \pm en(\vec{E} + \vec{v_\perp} \times \vec{B_0}) - KT\nabla n$$

Since we are interested in a motion perpendicular to the magnetic field (*i.e.*, perpendicular to z-axis), let us resolve this equation into x and y components:

$$mn\nu v_x = \pm en(E_x + v_y B_0) - KT\,\frac{\partial n}{\partial x}$$

and

$$mn\nu v_y = \pm en(E_y - v_x B_0) - KT\,\frac{\partial n}{\partial y}$$

Here, we have used $\vec{v_\perp} = v_x\hat{i} + v_y\hat{j}$, $\vec{B_0} = B_0\hat{k}$ and $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$. These equations may be rearranged as

$$v_x = \pm\frac{e}{m\nu}E_x - \frac{KT}{mn\nu}\,\frac{\partial n}{\partial x} \pm \frac{eB_0}{m\nu}\,v_y$$

and

$$v_y = \pm\frac{e}{m\nu}E_y - \frac{KT}{mn\nu}\,\frac{\partial n}{\partial y} \mp \frac{eB_0}{m\nu}\,v_x$$

After using mobility coefficient μ, diffusion coefficient D, and cyclotron frequency ω_c (in case of ion we shall take Ω_c), as

$$\mu = \frac{e}{m\nu} \qquad\qquad D = \frac{KT}{m\nu} \qquad\qquad \omega_c = \frac{eB_0}{m}$$

we get

$$v_x = \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} \pm \frac{\omega_c}{\nu}v_y \qquad\qquad v_y = \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} \mp \frac{\omega_c}{\nu}v_x$$

Substituting the expression for v_x in v_y, we get

$$v_y = \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} \mp \frac{\omega_c}{\nu}\left[\pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} \pm \frac{\omega_c}{\nu}v_y\right]$$

$$= \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \frac{\omega_c}{\nu}\mu E_x \pm \frac{\omega_c}{\nu}\frac{D}{n}\frac{\partial n}{\partial x} - \frac{\omega_c^2}{\nu^2}v_y$$

Using $\tau = \nu^{-1}$, we have

$$(1+\omega_c^2\tau^2)v_y = \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \frac{\omega_c}{\nu}\frac{e}{m\nu}E_x \pm \frac{\omega_c}{\nu}\frac{KT}{m\nu}\frac{1}{n}\frac{\partial n}{\partial x}$$

$$= \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \omega_c^2\tau^2\frac{E_x}{B_0} \pm \omega_c^2\tau^2\frac{KT}{e\,B_0}\frac{1}{n}\frac{\partial n}{\partial x} \tag{7.19}$$

Now, substitution of v_y in v_x gives

$$v_x = \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} \mp \frac{\omega_c}{\nu}\left[\pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} \pm \frac{\omega_c}{\nu}v_x\right]$$

$$= \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \frac{\omega_c}{\nu}\mu E_y \mp \frac{\omega_c}{\nu}\frac{D}{n}\frac{\partial n}{\partial y} - \frac{\omega_c^2}{\nu^2}v_x$$

Using $\tau = \nu^{-1}$, we have

$$(1+\omega_c^2\tau^2)v_x = \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \frac{\omega_c}{\nu}\frac{e}{m\nu}E_y \mp \frac{\omega_c}{\nu}\frac{KT}{m\nu}\frac{1}{n}\frac{\partial n}{\partial y}$$

$$= \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \omega_c^2\tau^2\frac{E_y}{B_0} \mp \omega_c^2\tau^2\frac{KT}{eB_0}\frac{1}{n}\frac{\partial n}{\partial y} \tag{7.20}$$

The last two terms in the equations (7.19) and (7.20) have the $\vec{E}\times\vec{B}_0$ and diamagnetic drifts

$$v_{Ex} = \frac{E_y}{B_0} \qquad\qquad v_{Dx} = \mp\frac{KT}{eB_0}\frac{1}{n}\frac{\partial n}{\partial y}$$

$$v_{Ey} = -\frac{E_x}{B_0} \qquad\qquad v_{Dy} = \pm\frac{KT}{eB_0}\frac{1}{n}\frac{\partial n}{\partial x}$$

By defining the perpendicular mobility and diffusion coefficients, respectively, as

$$\mu_\perp = \frac{\mu}{1+\omega_c^2\tau^2} \qquad\qquad D_\perp = \frac{D}{1+\omega_c^2\tau^2}$$

equation (7.20) can be expressed as

$$\begin{aligned} v_x &= \pm\mu_\perp E_x - \frac{D_\perp}{n}\frac{\partial n}{\partial x} + \frac{\omega_c^2\tau^2}{1+\omega_c^2\tau^2} v_{Ex} + \frac{\omega_c^2\tau^2}{1+\omega_c^2\tau^2} v_{Dx} \\ &= \pm\mu_\perp E_x - \frac{D_\perp}{n}\frac{\partial n}{\partial x} + \frac{v_{Ex}+v_{Dx}}{1+(\nu^2/\omega_c^2)} \end{aligned} \tag{7.21}$$

Similarly, equation (7.19) can be expressed as

$$v_y = \pm\mu_\perp\, E_y - \frac{D_\perp}{n}\frac{\partial n}{\partial y} + \frac{v_{Ey}+v_{Dy}}{1+(\nu^2/\omega_c^2)} \tag{7.22}$$

Equations (7.21) and (7.22) can be written together as

$$v_\perp = \pm\mu_\perp E - D_\perp\frac{\nabla n}{n} + \frac{v_E+v_D}{1+(\nu^2/\omega_c^2)} \tag{7.23}$$

Although $\mu_e > \mu_i$ and $D_e > D_i$, but $\mu_{e\perp} < \mu_{i\perp}$ and $D_{e\perp} < D_{i\perp}$, and therefore, $v_{e\perp} < v_{i\perp}$. Equation (7.23) shows that the perpendicular velocity of either species has two parts: (i) There are usual v_E and v_D drifts, perpendicular to the electric field and the density gradient, respectively. These drifts are slowed down by a factor $1+(\nu^2/\omega_c^2)$ due to collisions with neutral particles. In absence of collisions, the drag factor $1+(\nu^2/\omega_c^2)$ becomes unity. (ii) There are mobility and diffusion drifts parallel to the electric field and the density gradient, respectively. These drifts are similar to the case in absence of the applied magnetic field, but the coefficients are reduced by a factor $(1+\omega_c^2\tau^2)$. When $(\omega_c^2\tau^2) << 1$, the magnetic field does not have significant effect on the mobility and diffusion drifts. When $(\omega_c^2\tau^2) >> 1$, the magnetic field significantly retards the mobility and diffusion drifts across the magnetic field.

7.4.1 Ambipolar diffusion across the magnetic field

We have seen that mobility and diffusion coefficients are anisotropic in the presence of a magnetic field. When they are μ and D, respectively, along the magnetic field, then perpendicular to the magnetic field, they

are reduced by a factor $(1+\omega_c^2\tau^2)$. Therefore, ambipolar diffusion is not as forward as in absence of the magnetic field. Consider the particle flux perpendicular to the field lines (Figure 7.7). Both the ions and electrons would have their fluxes parallel and perpendicular to the lines of force. Ordinarily, $\Phi_{e\perp}$ is smaller than $\Phi_{i\perp}$ (as $v_{e\perp} < v_{i\perp}$), and therefore, a transverse electric field is set up so as to accelerate the electron diffusion and to retard the ion diffusion.

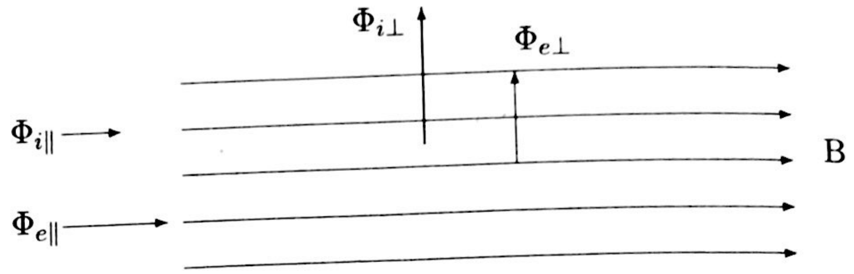

Figure 7.7: Shows the parallel and perpendicular particle fluxes in a magnetic field.

Although the total diffusion must be ambipolar, the perpendicular part of the ion and electron diffusions need not to be ambipolar. The ions can diffuse out primarily across the magnetic field whereas the electrons along the magnetic field. Each specie then diffuses across the magnetic field at a different rate.

Mathematically, we have to solve simultaneously the equation of continuity (Equation 7.4) both for ions and electrons, where the divergences $\nabla.\vec{\Phi}_j$ of the flux must be set equal to each other. Separating $\nabla.\vec{\Phi}_j$ into the components parallel and perpendicular to the magnetic field, we have

$$\nabla.\vec{\Phi}_i = \nabla_\perp.(\mu_{i\perp} n \vec{E}_\perp - D_{i\perp}\nabla_\perp n) + \frac{\partial}{\partial z}\left(\mu_{iz} n \vec{E}_z - D_{iz}\frac{\partial n}{\partial z}\right)$$

and

$$\nabla.\vec{\Phi}_e = \nabla_\perp.(-\mu_{e\perp} n \vec{E}_\perp - D_{e\perp}\nabla_\perp n) + \frac{\partial}{\partial z}\left(-\mu_{ez} n \vec{E}_z - D_{ez}\frac{\partial n}{\partial z}\right)$$

The equation obtained by putting $\nabla.\vec{\Phi}_i = \nabla.\vec{\Phi}_e$ cannot be easily separated into one dimensional equations. Moreover, the answer depends sensitively on the boundary conditions at the end of the field lines.

7.5 Collisions in a fully ionized plasma

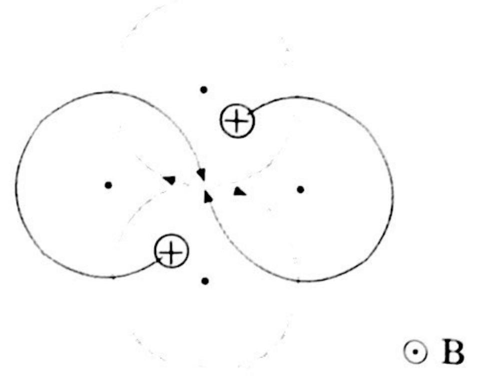

Figure 7.8: Guiding centers of two like particles after collision shift by 90^o.

When the plasma is fully ionized, *i.e.*, it has only positive ions and electrons, then all the collisions are between the charged particles. A collision between two like particles (ion-ion or electron-electron collision) is distinct from the collision between two unlike particles (ion-electron collision). For collision of two like particles, if it is a head-on collision, they emerge with their velocities reversed and their guiding-centers remain in the same place. The worst case can be when the particles collide at 90^o, then their velocities are changed by 90^o in direction, and the guiding-centers are shifted. However, the center of mass of the two guiding-centers remains stationary. Thus, the collisions between like particles give rise to very little diffusion. (This situation is to be contrasted with the collision of a charged particle with a neutral particle where diffusion was quite appreciable.)

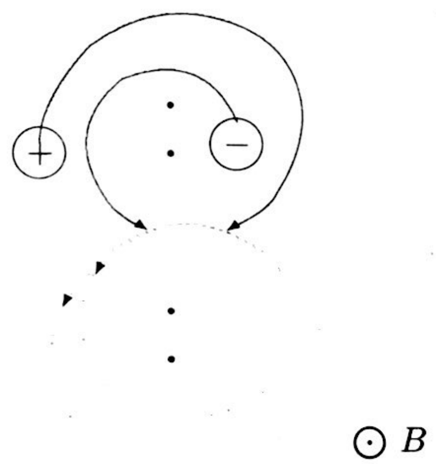

Figure 7.9: Guiding centers of two oppositely charged particles after collision shift by 180^o.

When two particles of opposite charge (positive ion with electron) collide, the worst case can now be the 180^o collision in which the particles emerge with their velocities reversed. As they must continue to gyrate in the proper sense about the field lines, guiding centers of both the particles shift in the same direction. Hence, the collision of two opposite charges give rise to diffusion.

7.5.1 Plasma resistivity

The fluid equations of motion for the charged particles including the collision can be now written as

$$m_i n \frac{d\vec{v}_i}{dt} = en(\vec{E} + \vec{v}_i \times \vec{B}) - \nabla p_i - \nabla \pi_i + \vec{P}_{ie}$$

and

$$m_e n \frac{d\vec{v}_e}{dt} = -en(\vec{E} + \vec{v}_e \times \vec{B}) - \nabla p_e - \nabla \pi_e + \vec{P}_{ei} \tag{7.24}$$

where the term $\vec{P}_{ie}$ represents the momentum gained by the ion fluid due to collisions with electrons and the term $\vec{P}_{ei}$ represents the momentum gained by the electron fluid due to collisions with ions. The stress tensor has been divided into the isotropic component p and the anisotropic component π. The collision between two like particles does not contribute much to the diffusion and we can ignore the term $\nabla\pi$. The terms $\vec{P}_{ie}$ and $\vec{P}_{ei}$ represent the friction between the two fluids and the conservation of momentum gives

$$\vec{P}_{ie} = -\vec{P}_{ei}$$

In terms of the collision frequency, the term $\vec{P}_{ei}$ can be written as

$$\vec{P}_{ei} = m_e n(\vec{v}_i - \vec{v}_e)\nu_{ei} \tag{7.25}$$

On the physical ground, $\vec{P}_{ei}$ should directly depend on the charges of the colliding particles, should be proportional to the densities of the particles, and proportional to the relative velocity $(\vec{v}_i - \vec{v}_e)$ of the particles. Hence, we can write

$$\vec{P}_{ei} = \eta e^2 n^2(\vec{v}_i - \vec{v}_e) \tag{7.26}$$

where η is a constant of proportionality. Comparison of equations (7.25) and (7.26) gives

$$\nu_{ei} = \frac{ne^2}{m_e}\eta$$

The constant η is interpreted as the *specific resistivity* of the plasma.

7.5.2 Physical meaning of η

For an electric field $\vec{E}$ existing in a plasma, the current is due to the motion of electrons. If the applied magnetic field $\vec{B}_0 = 0$ and $KT_e = 0$ so that $\nabla p_e = 0$. Then in the steady state, the electron equation(7.24) gives

$$0 = -en\vec{E} + \vec{P}_{ei} \qquad\qquad en\vec{E} = \vec{P}_{ei} \tag{7.27}$$

Since the current density $\vec{j} = en(\vec{v}_i - \vec{v}_e)$, equation (7.26) gives

$$\vec{P}_{ei} = \eta en\vec{j} \tag{7.28}$$

Comparison of equations (7.27) and (7.28) gives

$$\vec{E} = \eta\vec{j}$$

It is simply the Ohm's law, showing that η is just the specific resistivity.

7.6 Single-fluid MHD equations

Now, we want to discuss the problem of diffusion in a fully ionized plasma. As the dissipative term $\vec{P}_{ei}$ depends on the relative velocity $(\vec{v}_i - \vec{v}_e)$, it would be convenient to work with a linear combination of the ion and electron equations where $(\vec{v}_i - \vec{v}_e)$ is unknown rather than with the individual equations separately where $\vec{v}_i$ and $\vec{v}_e$ are unknown independently. By dealing with the equations for ions and electrons separately, we have so far dealt with the two inter-penetrating fluids. The linear combination of the equations for ions and electrons would describe the plasma as a single fluid, and the equations dealing with the single fluid are known as the *equations of magnetohydrodynamics* (MHD).

Let us consider a singly ionized plasma (hydrogen plasma) and define the mean density ρ, mean velocity $\vec{v}$, and current density $\vec{j}$ as

$$\rho \equiv n_i m_i + n_e m_e \approx n(m_i + m_e) \tag{7.29}$$

$$\vec{v} \equiv \frac{1}{\rho}(n_i m_i \vec{v}_i + n_e m_e \vec{v}_e) \approx \frac{m_i \vec{v}_i + m_e \vec{v}_e}{m_i + m_e} \tag{7.30}$$

$$\vec{j} \equiv e(n_i \vec{v}_i - n_e \vec{v}_e) \approx ne(\vec{v}_i - \vec{v}_e) \tag{7.31}$$

In the equations of motion for ion and electron, we shall now include the gravitational force in order to represent any non-electromagnetic force. The equations of motion for ion and electron can be written as

$$m_i n \frac{d\vec{v}_i}{dt} = en(\vec{E} + \vec{v}_i \times \vec{B}) - \nabla p_i + m_i n \vec{g} + \vec{P}_{ie} \tag{7.32}$$

and

$$m_e n \frac{d\vec{v}_e}{dt} = -en(\vec{E} + \vec{v}_e \times \vec{B}) - \nabla p_e + m_e n \vec{g} + \vec{P}_{ei} \tag{7.33}$$

Here, we have not accounted for the anisotropic component π as it would not introduce much error if the Larmor radius is much smaller than the scale length over which the various quantities change. For linear treatment of the problem, we have to neglect the term $(\vec{v} \cdot \nabla)\vec{v}$. Though this simplification is more difficult to justify. However, we may say that $\vec{v}$ is assumed to be so small that this quadratic term is negligible. Addition of equations (7.32) and (7.33) gives

$$n\frac{\partial}{\partial t}(m_i \vec{v}_i + m_e \vec{v}_e) = en(\vec{v}_i - \vec{v}_e) \times \vec{B} - \nabla(p_i + p_e) + n(m_i + m_e)\vec{g} \tag{7.34}$$

Here, we have used $\vec{P}_{ei} = -\vec{P}_{ie}$. Defining total pressure $p = p_i + p_e$ and using equations (7.29) – (7.31) in (7.34), we get

$$n\frac{\partial}{\partial t}[\vec{v}\,(m_i + m_e)] = \vec{j} \times \vec{B} - \nabla p + \rho\, \vec{g}$$

$$\rho\frac{\partial \vec{v}}{\partial t} = \vec{j} \times \vec{B} - \nabla p + \rho\, \vec{g} \tag{7.35}$$

This is a single fluid equation of motion describing the mass flow. Notice that the electric field does not appear explicitly because the fluid is neutral.

Now, let us multiply equation (7.32) by m_e and equation (7.33) by m_i and subtract the latter from the former to get

$$m_i m_e n \frac{\partial}{\partial t}(\vec{v}_i - \vec{v}_e) = en(m_i + m_e)\,\vec{E} + en(m_e \vec{v}_i + m_i \vec{v}_e) \times \vec{B}$$

$$-m_e \nabla p_i + m_i \nabla p_e - (m_i + m_e)\vec{P}_{ei} \tag{7.36}$$

Using equations (7.26), (7.29) and (7.31) in (7.36) , we get

$$m_i m_e n \frac{\partial}{\partial t}\left(\frac{\vec{j}}{ne}\right) = e\rho \vec{E} + en(m_e \vec{v}_i + m_i \vec{v}_e) \times \vec{B} - m_e \nabla p_i$$

$$+ m_i \nabla p_e - (m_i + m_e)\eta e^2 n^2 (\vec{v}_i - \vec{v}_e)$$

$$\frac{m_i m_e n}{e} \frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right) = e\rho \vec{E} + en(m_e \vec{v}_i + m_i \vec{v}_e) \times \vec{B} - m_e \nabla p_i$$

$$+ m_i \nabla p_e - \rho \eta e \vec{j} \tag{7.37}$$

Now, we can write

$$m_e \vec{v}_i + m_i \vec{v}_e = m_i \vec{v}_i + m_e \vec{v}_e + m_i(\vec{v}_e - \vec{v}_i) + m_e(\vec{v}_i - \vec{v}_e)$$

$$= (m_i + m_e)\vec{v} - (m_i - m_e)\frac{\vec{j}}{ne} = \frac{\rho}{n}\vec{v} - (m_i - m_e)\frac{\vec{j}}{ne} \tag{7.38}$$

Using equation (7.38) in (7.37), we get

$$\frac{m_i m_e n}{e} \frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right) = e\rho \vec{E} + en\left[\frac{\rho}{n}\vec{v} - (m_i - m_e)\frac{\vec{j}}{ne}\right] \times \vec{B}$$

$$- m_e \nabla p_i + m_i \nabla p_e - \rho \eta e \vec{j}$$

After dividing by $e\rho$, this equation can be rearranged as

$$\vec{E} + \vec{v} \times \vec{B} - \eta \vec{j} = \frac{1}{e\rho}\left[\frac{m_i m_e n}{e} \frac{\partial}{\partial t}\left(\frac{\vec{j}}{n}\right) + (m_i - m_e)\,\vec{j} \times \vec{B}\right.$$

$$\left. + m_e \nabla p_i - m_i \nabla p_e\right]$$

For the slow motion, the term $\partial/\partial t$ can be neglected. In the limit $m_e/m_i \to 0$, we have

$$\vec{E} + \vec{v} \times \vec{B} - \eta \vec{j} = \frac{1}{e\rho}(m_i \, \vec{j} \times \vec{B} - m_i \nabla p_e)$$

$$= \frac{1}{en}(\vec{j} \times \vec{B} - \nabla p_e) \tag{7.39}$$

This is our second equation, called the generalized Ohm's law, for the single fluid plasma. The term $\vec{j} \times \vec{B}$ is known as the *Hall current.* This often happens that the terms on the right side of equation (7.39) are negligible, then the Ohm's law is

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j} \tag{7.40}$$

The equations of continuity for the ion and electron are

$$\frac{\partial n_i}{\partial t} + \nabla.(n_i \vec{v}_i) = 0 \qquad \text{and} \qquad \frac{\partial n_e}{\partial t} + \nabla.(n_e \vec{v}_e) = 0 \tag{7.41}$$

On multiply the first equation of (7.41) by m_i and the second by m_e and adding we get

$$\frac{\partial}{\partial t}(n_i m_i + n_e m_e) + \nabla.(n_i m_i \vec{v}_i + n_e m_e \vec{v}_e) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \vec{v}) = 0 \tag{7.42}$$

On multiply the first equation of (7.41) by e and the second by $-e$ and adding we get

$$\frac{\partial}{\partial t}(n_i e - n_e e) + \nabla.[e(n_i \vec{v}_i - n_e \vec{v}_e)] = 0$$

$$\frac{\partial \sigma}{\partial t} + \nabla. \, \vec{j} = 0 \tag{7.43}$$

where $\sigma = e(n_i - n_e)$ is the charge density. The equations (7.35), (7.40), (7.42) and (7.43) form a complete set of MHD equations. This set of equations along with Maxwell equations is used to describe the equilibrium state of a plasma. These MHD equations have been extensively used in astrophysics and in cosmic electrodynamics.

7.7 Diffusion in a fully ionized plasma

In absence of the gravity, the steady state plasma is given by

$$\vec{j} \times \vec{B} = \nabla p \qquad \text{and} \qquad \vec{E} + \vec{v} \times \vec{B} = \eta \vec{j} \tag{7.44}$$

The perpendicular component can be obtained by taking cross product of second equation of (7.44) with $\vec{B}$

$$\vec{E} \times \vec{B} + [(\vec{v_\perp} + \vec{v_\parallel}) \times \vec{B}] \times \vec{B} = \eta\, \vec{j} \times \vec{B}$$

$$\vec{E} \times \vec{B} + (\vec{v_\perp} \times \vec{B}) \times \vec{B} = \eta \nabla p$$

$$\vec{E} \times \vec{B} + \vec{B}\,(\vec{v}_\perp \cdot \vec{B}) - \vec{v}_\perp\,(\vec{B} \cdot \vec{B}) = \eta \nabla p$$

$$\vec{E} \times \vec{B} - \vec{v}_\perp\, B^2 = \eta \nabla p$$

$$\vec{v}_\perp = \frac{\vec{E} \times \vec{B}}{B^2} - \frac{\eta}{B^2}\,\nabla p \tag{7.45}$$

In equation (7.45), the first term is just $\vec{E} \times \vec{B}$ drift of both the ion and electron together, whereas the second term is the diffusion velocity in the direction opposite to ∇p. For the axisymmetric cylindrical plasma in which $\vec{E}$ and ∇p are in the radial direction, we have

$$\vec{E} = E_r \hat{r} \qquad \nabla p = \frac{\partial p}{\partial r}\hat{r} \qquad \vec{B} = B\hat{z}$$

$$\frac{\vec{E} \times \vec{B}}{B^2} = -\frac{E_r B}{B^2}\,\hat{\theta} = -\frac{E_r}{B}\,\hat{\theta}$$

Thus, the perpendicular components are

$$v_\theta = -\frac{E_r}{B} \qquad v_r = -\frac{\eta}{B^2}\frac{\partial p}{\partial r}$$

Here, radial component of the velocity would contribute to the diffusion. Thus, the flux associated with the diffusion is

$$\Phi = n_i \vec{v}_{\perp i} + n_e \vec{v}_{\perp e}$$

$$= -n_i \frac{\eta}{B^2} \nabla p_i - n_e \frac{\eta}{B^2} \nabla p_e = -n \frac{\eta K T_i}{B^2} \nabla n - n \frac{\eta K T_e}{B^2} \nabla n$$

$$= -\frac{\eta n (KT_i + KT_e)}{B^2} \nabla n = -D_\perp \nabla n$$

This can be interpreted as the Fick's law with the diffusion coefficient

$$D_\perp = \frac{\eta n (KT_i + KT_e)}{B^2}$$

This is generally known as the *classical diffusion coefficient* for a fully ionized plasma.

7.7.1 Solution of diffusion equation

In the diffusion equation

$$\Phi = -D_\perp \nabla n$$

the diffusion coefficient

$$D_\perp = \frac{\eta n (KT_i + KT_e)}{B^2}$$

is not constant. Let us take $T_i = T_e = T$ and assume that $\alpha = \eta KT/B^2$ is constant. Thus, $D_\perp = 2n\alpha$, and the equation of continuity is

$$\frac{\partial n}{\partial t} = -\nabla.\Phi = \nabla.(D_\perp \nabla n) = \nabla.(2n\alpha \nabla n)$$

$$= \alpha \nabla.(2n \nabla n) = \alpha \nabla^2 n^2 \tag{7.46}$$

This is a non-linear equation. It can however be solved partly analytically and partly by numerical techniques. On applying the method of the separation of variables to equation (7.46), we have

$$n(\vec{r}, t) = T(t)\ S(\vec{r}) \tag{7.47}$$

Using equation (7.47) in (7.46), we get

$$S\ \frac{\partial T}{\partial t} = \alpha T^2 \nabla^2 S^2 \qquad \frac{1}{T^2} \frac{\partial T}{\partial t} = \frac{\alpha}{S}\ \nabla^2 S^2 \tag{7.48}$$

In equation (7.48), the left side depends on t whereas the right side depends on $\vec{r}$. Hence, the equation can only be satisfied when each side

is equal to some constant, say $-1/\tau$. Thus, for the left side of equation (7.48), we have

$$\frac{1}{T^2}\frac{dT}{dt} = -\frac{1}{\tau} \qquad\qquad \frac{1}{T} = \frac{1}{T_0} + \frac{t}{\tau}$$

where $T = T_0$ at $t = 0$. It shows that density of the charged particles decreases exponentially with time. For the right side of equation (7.48), we have

$$\nabla^2 S^2 = -\frac{1}{\alpha\tau} S$$

This equation cannot be solved analytically, but through the numerical techniques. After knowing S and T, we can find out n.

7.8 Problems and questions

1. Discuss the process of ambipolar diffusion of various species in a weakly ionized plasma in absence of a magnetic field.

2. Discuss the process of diffusion of various species in a weakly ionized plasma across the magnetic field. Comment on the ambipolar diffusion across the magnetic field.

3. Discuss the phenomena of diffusion and recombination in a plasma. Show that the diffusion is more effective in comparison to the recombination.

4. Derive the single fluid MHD equations for a fully ionized plasma.

5. Derive an expression for diffusion coefficient for a fully ionized plasma.

6. Discuss about a fully ionized gas.

7. Write short notes on the following

   (i) Recombination process in the plasma

   (ii) Collision of two charged particles in a fully ionized plasma

8 Equilibrium and Stability

8.1 Introduction

If we talk about the individual particles moving in a plasma, it looks easy to design a magnetic field to confine a collision-less plasma. The main requirement in the design of a magnetic field is that the field lines do not hit the vacuum wall. Hence, the field is designed in such a way that all the particle drifts are made parallel to the walls. Dealing with individual particles is quite different from that with a plasma, as the plasma itself generates internal fields which obviously affect its motion. For example, a gradient in charge distribution can create an electric field $\vec{E}$ which can cause $\vec{E} \times \vec{B}$ drifts; the currents in the plasma can create the magnetic fields and hence a gradient in the magnetic field, causing a drift. The drifts may move the particles to the wall. Hence, it is not easy to understand whether the magnetic field designed to confine individual particles can also confine a plasma.

The problem of plasma confinement can be studied in two parts: (i) the problem of equilibrium and (ii) the problem of stability. The difference between the equilibrium and stability can be understood in the following manner. When all the forces acting on a system are balanced, the system is said to be in an equilibrium state. This equilibrium state is stable or unstable can be decided by giving small perturbations to the system from the equilibrium state. The equilibrium is said to be stable or unstable according to whether the small perturbations are damped or amplified.

When we talk about a plasma, the situation is more complicated in comparison to that for the mechanical systems. Figure 8.1 shows four cases, A, B, C, D of one-dimensional systems where variation of potential energy V as a function of x is shown. The equilibrium position in

each case is represented by x_0. When a particle in the equilibrium position is given a small displacement, in A and C cases, the particle will move to and fro about the equilibrium position and slowly will come to rest at its equilibrium position. In the other two cases, B and D, the particle will never come back to its equilibrium position. Thus, A and C are the examples of stable equilibrium whereas B and D are of unstable equilibrium. Further, in the case C, only small oscillations are permissible. In a plasma, an equilibrium requires balancing the forces on each fluid element. From the dealing point of view, the problem of the equilibrium is more difficult than that the stability. A reason for it can be understood as follows. For small perturbations, the problem of stability can be linearized just as in the case of plasma waves. On the other hand, the problem of equilibrium is non-linear, and cannot be linearized by any means.

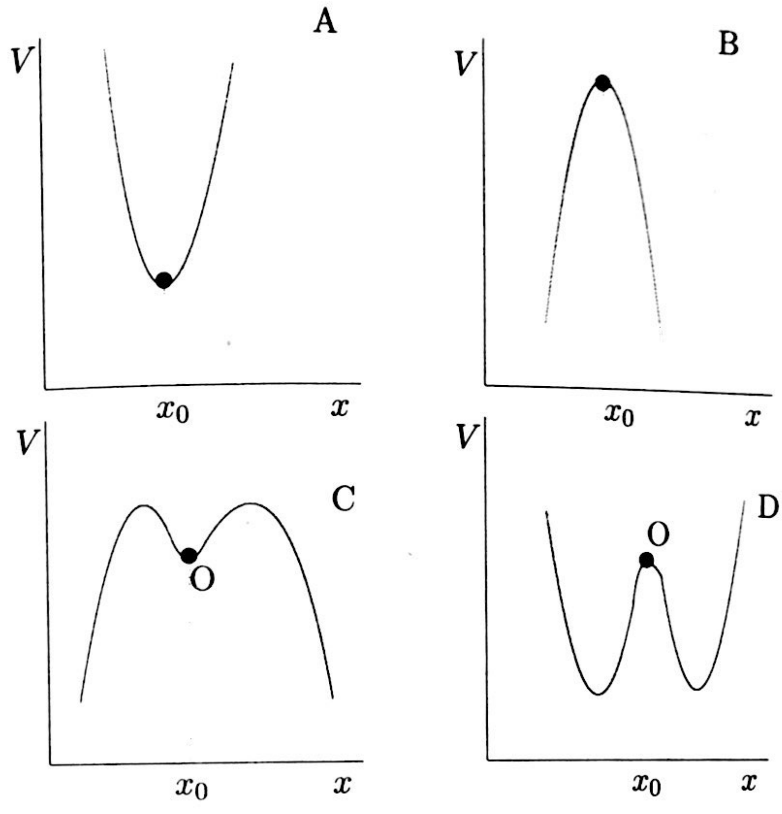

Figure 8.1: Four cases A, B, C and D for variation of potential energy V as a function of x. Here, x_0 represents the equilibrium position.

8.2 Hydromagnetic equilibrium

The single fluid equation of motion describing the mass flow is

$$\rho \frac{\partial \vec{v}}{\partial t} = \vec{j} \times \vec{B} - \nabla p + \rho \vec{g}$$

For a steady state $\partial/\partial t = 0$ and $\vec{g} = 0$, we get

$$\nabla p = \vec{j} \times \vec{B} \tag{8.1}$$

and one of the Maxwell's equations is

$$\nabla \times \vec{B} = \mu_0 \vec{j} \tag{8.2}$$

The simple equation (8.1) can be used to make several observations. Some of them are as the following.

(i) Equation (8.1) shows a balance between Lorentz force and a force due to the pressure gradient. In order to understand it, let us consider a cylindrical plasma where ∇p is directed radially inward (Figure 8.2). Thus, there would be outward force of expansion. In order to counteract this force of expansion, there must be an azimuthal current in the direction shown in the figure. The magnitude of the current required for having equilibrium can be estimated by taking the cross product of equation (8.1) with $\vec{B}$ as

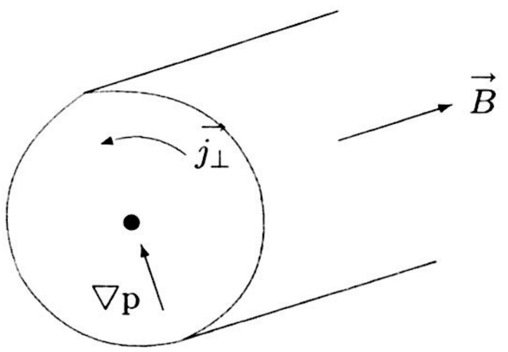

Figure 8.2: In the steady state, the pressure-gradient force is balanced by the $\vec{j}_\perp \times \vec{B}$ force of the diamagnetic current.

$$\vec{B} \times \nabla p = \vec{B} \times (\vec{j} \times \vec{B})$$

$$= \vec{B} \times (\vec{j}_\perp \times \vec{B}) = \vec{j}_\perp (\vec{B} \cdot \vec{B}) - \vec{B} (\vec{B} \cdot \vec{j}_\perp) = \vec{j}_\perp B^2$$

Hence,

$$\vec{j}_\perp = \frac{\vec{B} \times \nabla p}{B^2} = \frac{\vec{B} \times \nabla (p_i + p_e)}{B^2} = (KT_i + KT_e)\, \frac{\vec{B} \times \nabla n}{B^2}$$

It is the same as the diamagnetic current, obtained earlier. It is generated by the ∇p force across $\vec{B}$ and is sufficient to balance the forces on each element of fluid and to stop the motion.

(ii) Equation (8.1) shows that both $\vec{j}$ and $\vec{B}$ are perpendicular to ∇p. This situation is possible in the form of a rather complicated geometry. Consider a toroidal plasma having a smooth radial density gradient so that the surfaces of constant density (that is constant pressure) are nested tori (Figure 8.3)

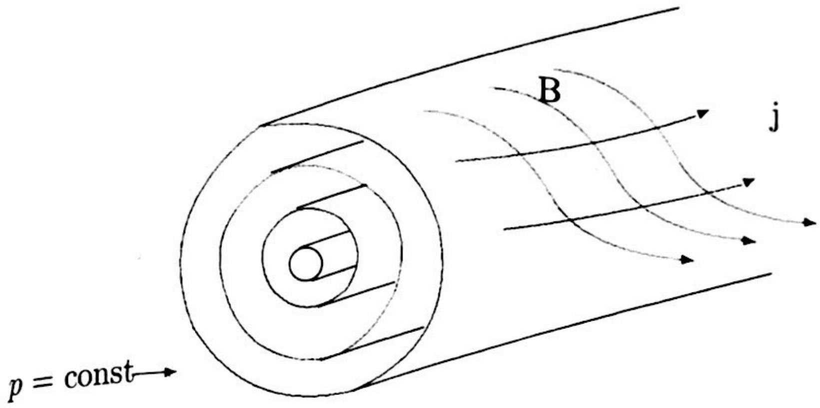

Figure 8.3: Both the vectors $\vec{j}$ and $\vec{B}$ lie on the constant-pressure surfaces.

Since $\vec{j}$ and $\vec{B}$ are perpendicular to ∇p, they must lie on the surface of constant p. The field lines and direction of current may be twisted in such a way that they must not cross the constant-p surfaces.

(iii) Consider the component of equation (8.1) along the magnetic field along the field lines. It gives

$$\frac{\partial p}{\partial s} = 0$$

where s is the coordinate measured along the field line. The right side is zero as the $\vec{j} \times \vec{B}$ is always perpendicular to $\vec{B}$. For constant temperature T, this equation can be written as

$$KT\, \frac{\partial n}{\partial s} = 0$$

showing that in the hydromagnetic equilibrium, the density is constant along a field line. One may be doubt about this conclusion. In order to understand it, let us consider a plasma injected into a magnetic mirror (Figure 8.4). As the plasma streams through, following the field lines, it expands and then contracts showing that the density is clearly not constant along a field line. This situation however does not satisfy the condition of a static equilibrium. The $(\vec{v}\,.\nabla)\vec{v}$ term, which we neglected along the way, does not vanish here. Hence, we must consider a static plasma with $\vec{v}=0$. In that situation, particles are trapped in the mirror in such a manner that there are more particles near the mid-plane than near the ends as the mirror ratio is large there. As the area of cross section is larger at the mid-plane than at the ends, the density of particles is constant along the field lines.

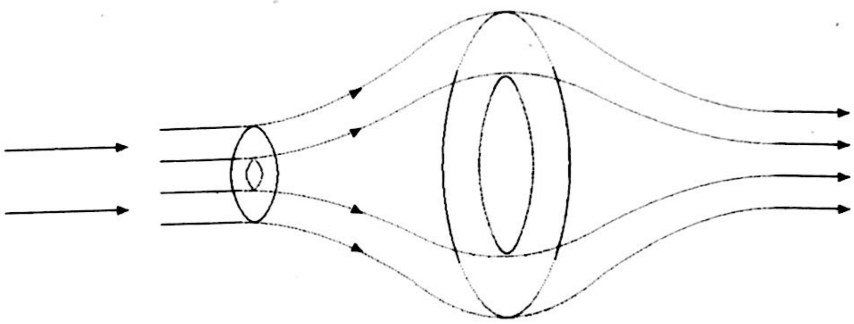

Figure 8.4: Expansion of plasma streams into a mirror.

8.3 Concept of β

On substituting value of $\vec{j}$ from equation (8.2) in (8.1), we get

$$\nabla p = \mu_0^{-1}\,(\nabla\times\vec{B})\times\vec{B} = \mu_0^{-1}\left[\left(\vec{B}\,.\nabla\right)\vec{B} - \frac{1}{2}\,\nabla B^2\right]$$

$$\nabla\left(p + \frac{B^2}{2\mu_0}\right) = \frac{1}{\mu_0}\left(\vec{B}\,.\nabla\right)\vec{B} \tag{8.3}$$

In many interesting cases, for example, for the axial cylindrical field,[1]

[1] We have

$\vec{B} = B\hat{z}$ $\qquad$ $\vec{B}\,.\nabla = B\,\dfrac{\partial}{\partial z}$ $\qquad$ $(\vec{B}\,.\nabla)\,\vec{B} = 0$

the right side vanishes; in many other cases, the right side is small. Thus, from equation (8.3), we have

$$p + \frac{B^2}{2\mu_0} = \text{constant} \tag{8.4}$$

The term $B^2/2\mu_0$ is the magnetic field pressure. Equation (8.4) shows that the sum of the particle pressure and magnetic field pressure is constant. Therefore, the magnetic field must be low where the density is high and vice versa. Inside a plasma, decrease of magnetic field is caused by the diamagnetic current. The measure of the diamagnetic effect is expressed in term of the ratio

$$\beta = \frac{p}{B^2/2\mu_0} = \frac{\text{particle pressure}}{\text{magnetic field pressure}}$$

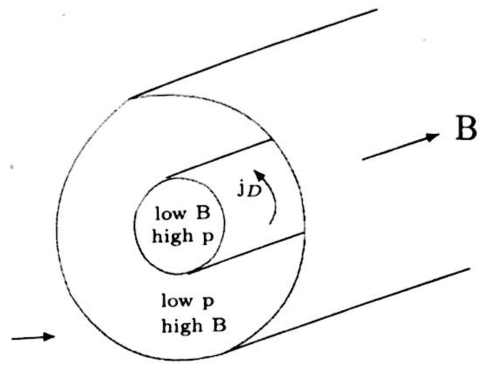

Figure 8.5: For a constant sum of magnetic and particle pressures in a finite-β plasma, the diamagnetic current significantly decreases the magnetic field.

So far we have implicitly considered low-β plasma, where the value of β is between 10^{-3} and 10^{-6}. For low value of β, the diamagnetic effect is very small and thus we considered a uniform magnetic field $\vec{B_0}$ in the treatment of plasma waves.

When β is high, the local value of the magnetic field can be reduced by the plasma. High-β plasma are found in space and is in the MHD energy conversion research.

In principle, one can have a $\beta = 1$ plasma where the diamagnetic current generates a magnetic field exactly equal and opposite to an externally generated uniform field.

8.4 Diffusion of magnetic field into a plasma

The problem of diffusion of magnetic field into a plasma is quite common in astrophysics. Consider a situation where the two regions, one with a plasma but no magnetic field and the second with a magnetic field but no plasma, are separated by a common boundary (Figure 8.6). When the plasma has no resistivity, *i.e.*, it is superconductor, the magnetic lines force cannot penetrate the plasma. The motion of plasma thus pushes the field lines and can bend and twist them.

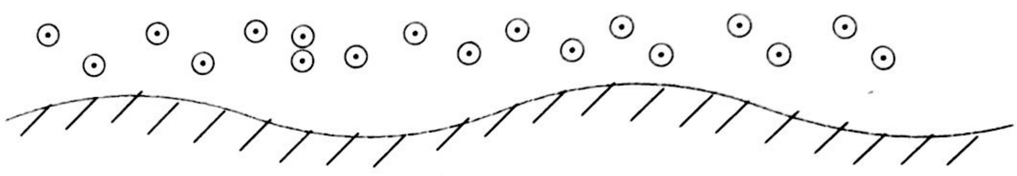

Figure 8.6: When the plasma is perfectly conducting, regions of plasma and magnetic field are separated by a sharp boundary. Currents on the surface exclude the field from the plasma.

If the resistivity of the plasma is finite, the plasma can move through the field lines and vice versa. This diffusion however takes a certain amount of time. When the motions are slow enough, the field lines need not be distorted by the motion of plasma. The diffusion time can easily be calculated from the equations

$$\nabla\times \vec{E} = -\dot{\vec{B}} \qquad\qquad \vec{E} + \vec{v}\times\vec{B} = \eta\,\vec{j}$$

For convenience, let us assume that the plasma is at rest and the field lines are moving into it. Then $\vec{v} = 0$ and we have $\vec{E} = \eta\,\vec{j}$ and therefore,

$$\frac{\partial \vec{B}}{\partial t} = -\nabla\times\vec{E} = -\nabla\times(\eta\,\vec{j}) \tag{8.5}$$

Using equation (8.2) in (8.5) we get

$$\frac{\partial \vec{B}}{\partial t} = -\frac{\eta}{\mu_0}\nabla\times(\nabla\times\vec{B}) = -\frac{\eta}{\mu_0}[\nabla(\nabla.\vec{B}) - \nabla^2\vec{B}]$$

Since $\nabla . \vec{B} = 0$, the diffusion of the magnetic field through the plasma is expressed as

$$\frac{\partial \vec{B}}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 \vec{B} \tag{8.6}$$

Solution of equation (8.6)

Equation (8.6) can be solved by the method of the separation of variables

$$\vec{B}(\vec{r}, t) = T(t)\, \vec{S}(\vec{r}) \tag{8.7}$$

Using equation (8.7) in (8.6), we get

$$\vec{S} \frac{\partial T}{\partial t} = \frac{\eta}{\mu_0} T \nabla^2 \vec{S} \qquad \frac{1}{T}\frac{\partial T}{\partial t} = \frac{\eta}{\vec{S}\,\mu_0} \nabla^2 \vec{S} \tag{8.8}$$

In equation (8.8), the left side depends on t whereas the right side depends on $\vec{r}$. Hence, the equation can only be satisfied when each side is equal to some constant, say $-1/\tau$. Thus, for the left side of equation (8.8), we have

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{1}{\tau} \qquad \frac{\mathrm{d}T}{T} = -\frac{1}{\tau}\,\mathrm{d}t$$

so that

$$T = T_0\, \mathrm{e}^{-t/\tau} \tag{8.9}$$

where $T = T_0$ at $t = 0$. It shows that density of the charged particles decreases exponentially with time. For the right-side of equation (8.8), we have

$$\frac{\eta}{\mu_0 \vec{S}} \nabla^2 \vec{S} = -\frac{1}{\tau} \qquad \nabla^2 \vec{S} = -\frac{\mu_0}{\eta\tau}\, \vec{S} \tag{8.10}$$

Equation (8.10) can be solved for various geometries of the container of plasma.

Diffusion in a slab

For a slab, for variation in one-dimension, equation (8.10) can be written as

$$\frac{\mathrm{d}^2 \vec{S}}{\mathrm{d}x^2} = -\frac{\mu_0}{\eta\tau}\, \vec{S}$$

General solution of this equation is

$$\vec{S}=\vec{A}\cos\left(\frac{x\sqrt{\mu_0}}{\sqrt{\eta\tau}}\right)+\vec{B}\sin\left(\frac{x\sqrt{\mu_0}}{\sqrt{\eta\tau}}\right)$$

where A and B are constants. Since we expect solution to be symmetrical about the $x=0$ plane, we can ignore the sine term and thus we have

$$\vec{S}=\vec{A}\cos\left(\frac{x\sqrt{\mu_0}}{\sqrt{\eta\tau}}\right)$$

If we assume that the magnetic field is zero at the walls $x=\pm l$ of the container, we have

$$0=\vec{A}\cos\left(\frac{l\sqrt{\mu_0}}{\sqrt{\eta\tau}}\right)$$

and thus,

$$\frac{l\sqrt{\mu_0}}{\sqrt{\eta\tau}}=(2n+1)\,\frac{\pi}{2}$$

where n is an integer. We account for the simplest case of $n=0$, and hence,

$$\vec{S}=\vec{A}\cos\left(\frac{x\pi}{2l}\right) \tag{8.11}$$

Using equations (8.9) and (8.11) in (8.7), we get

$$\vec{B}\,(\vec{r},t)=T_0\,\mathrm{e}^{-t/\tau}\,\vec{A}\cos\left(\frac{x\pi}{2l}\right)$$

$$=\vec{B_0}\;\mathrm{e}^{-t/\tau}\cos\left(\frac{x\pi}{2l}\right)$$

where $\vec{B_0}\;(=T_0\,\vec{A})$ is the magnetic field at $t=0$ and $x=0$.

Diffusion in a cylinder

For a cylinder, equation (8.10) can be written as

$$\frac{\mathrm{d}^2\vec{S}}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}\vec{S}}{\mathrm{d}r}+\frac{\mu_0}{\eta\tau}\,\vec{S}=0$$

$$r^2\,\frac{\mathrm{d}^2\vec{S}}{\mathrm{d}r^2}+r\,\frac{\mathrm{d}\vec{S}}{\mathrm{d}r}+\frac{\mu_0 r^2}{\eta\tau}\,\vec{S}=0 \tag{8.12}$$

Here, we are considering radial variation of the density of the charged particles. Let $r = x\sqrt{\eta\tau}/\sqrt{\mu_0}$. Then

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{\sqrt{\mu_0}}{\sqrt{\eta\tau}}$$

$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}r} = \frac{\mathrm{d}\vec{S}}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{\sqrt{\mu_0}}{\sqrt{\eta\tau}}\frac{\mathrm{d}\vec{S}}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2\vec{S}}{\mathrm{d}r^2} = \frac{\sqrt{\mu_0}}{\sqrt{\eta\tau}}\frac{\mathrm{d}^2\vec{S}}{\mathrm{d}x^2}\frac{\mathrm{d}x}{\mathrm{d}r} = \frac{\mu_0}{\eta\tau}\frac{\mathrm{d}^2\vec{S}}{\mathrm{d}x}$$

Using these relations in equation (8.12), we get

$$x^2\frac{\mathrm{d}^2\vec{S}}{\mathrm{d}x^2} + x\frac{\mathrm{d}\vec{S}}{\mathrm{d}x} + x^2\,\vec{S} = 0$$

This is the zeroth order Bessel equation and its solution is the Bessel function $J_0(x)$. Thus the solution of equation (8.12) is

$$\vec{S} = \vec{A}\,J_0(x) = \vec{A}\,J_0\left(\frac{r\sqrt{\mu_0}}{\sqrt{\eta\tau}}\right) \tag{8.13}$$

where $\vec{A}$ is a constant. Since, we expect the magnetic field to be nearly zero ($\vec{S} = 0$) at the surface of the cylinder $r = R$, we have

$$J_0\left(\frac{R\sqrt{\mu_0}}{\sqrt{\eta\tau}}\right) = 0$$

For the first zero of the Bessel function, we have

$$\frac{R\sqrt{\mu_0}}{\sqrt{\eta\tau}} = 2.4 \tag{8.14}$$

Using equations (8.14) in (8.13), we get

$$\vec{S} = \vec{A}\,J_0\left(\frac{2.4r}{R}\right) \tag{8.15}$$

Using equations (8.9) and (8.15) in (8.7), we get

$$\vec{B}\,(\vec{r}, t) = T_0\,\mathrm{e}^{-t/\tau}\,\vec{A}\,J_0\left(\frac{2.4r}{R}\right) = \vec{B}_0\;\mathrm{e}^{-t/\tau}J_0\left(\frac{2.4r}{R}\right)$$

where $\vec{B}_0$ $(= T_0\,\vec{A})$ is the magnetic field at $t = 0$ and $r = 0$.

Energy dissipation

In order to make a rough estimate, let L be the scale length of the spatial variation of $\vec{B}$. Then we can write

$$\frac{\partial \vec{B}}{\partial t} = \frac{\eta}{\mu_0}\nabla^2 \vec{B} \qquad\qquad \frac{B}{\tau} = \frac{\eta}{\mu_0 L^2}$$

where $\tau = \mu_0 L^2/\eta$ is the characteristic time for magnetic field penetration into a plasma.

As the field lines diffuse through the plasma, the magnetic field annihilates and the energy of the field converts into the ohmic heating of the plasma. The energy lost per m^3 in a time τ is $\eta j^2 \tau$. The value of j can be estimated from the Maxwell equation as

$$\mu_0 \vec{j} = \nabla \times \vec{B} \qquad\qquad j \approx \frac{B}{\mu_0 L}$$

Thus, the energy dissipation is

$$\eta j^2 \tau = \eta\left(\frac{B}{\mu_0 L}\right)^2 \frac{\mu_0 L^2}{\eta} = \frac{B^2}{\mu_0} = 2\,\frac{B^2}{2\mu_0}$$

8.5 Classification of instabilities

In the discussion of plasma waves, we assumed an unperturbed state which was one of the perfect thermodynamical equilibrium. In such a state of high entropy, there was no free energy available to excite plasma waves. We know consider states that are not in a perfect thermodynamical equilibrium. They are although in equilibrium in the sense that all forces are in balance and a time independent solution is possible. The free energy which is now available can cause the waves to be self-excited; the equilibrium is then an unstable one.

Depending on the type of free energy available, instabilities can be classified into four main categories.

1. **Streaming instability:** In this case, different species of plasma have drifts relative to one another. The drift energy is used to excite the waves and oscillation energy is obtained from the drift energy in the unperturbed state.

2. **Rayleigh-Taylor instability:** In this case, the density of plasma is not uniform and an external non-electromagnetic force is applied to the plasma. This external force drives the instability.

3. **Universal instability:** Even when there are no obvious driving forces such as an electric or a gravitational field, a plasma itself is not in a perfect thermodynamical equilibrium as long as it is confined. Here, the plasma pressure tends to make the plasma expand, and this expansion energy can drive an instability. In a finite plasma, this type of free energy is always present, and the resulting waves are called the universal instabilities.

4. **Kinetic instability:** In the fluid theory, the velocity distributions were assumed to be Maxwellian. The velocity distributions are in fact non-Maxwellian and thus, there is deviation from the thermodynamic equilibrium.

8.6 Two-stream instability

Consider a plasma having two species, positive ions and electrons. Let us consider a uniform plasma in which stationary ions form a background and the electrons are moving. Let the plasma is cold ($T_i = T_e = 0$) and there is no external magnetic field ($\vec{B_0}= 0$). Equations of motion and continuity of an ion are

$$m_i\left[\frac{\partial \vec{v_i}}{\partial t} + (\vec{v_i}\,.\nabla)\vec{v_i}\,\right] = e\vec{E} \qquad \text{and} \qquad \frac{\partial n_i}{\partial t} + \nabla.(n_i\vec{v_i}) = 0 \quad (8.16)$$

For linearisation, let us separate the variable parameters into two parts: (i) an equilibrium part, indicated by a subscript 0 and (ii) the perturbation part, oscillations, indicated by a subscript 1

$$n_i = n_{i0} + n_{i1} \qquad \vec{v_i} = \vec{v_{i0}} + \vec{v_{i1}} \qquad \vec{E} = \vec{E_0} + \vec{E_1}$$

We have $\vec{v_{i0}} = \vec{E_0} = 0$ and n_0 is homogeneous as well as independent of time, *i.e.*, $\nabla n_0 = \partial n_{i0}/\partial t = 0$. Then, linearisation of equations (8.16) gives

$$m_i\frac{\partial \vec{v_{i1}}}{\partial t} = e\vec{E_1} \qquad \text{and} \qquad \frac{\partial n_{i1}}{\partial t} + n_{i0}\nabla.\,\vec{v_{i1}}= 0 \qquad (8.17)$$

Equations of motion and continuity of an electron are

$$m_e\left[\frac{\partial \vec{v_e}}{\partial t} + (\vec{v_e}\,.\nabla)\,\vec{v_e}\right] = -e\vec{E} \qquad \text{and} \qquad \frac{\partial n_e}{\partial t} + \nabla.(n_e\vec{v_e}) = 0 \quad (8.18)$$

For linearisation, let us separate the variable parameters into two parts: (i) an equilibrium part, indicated by a subscript 0 and (ii) the perturbation part, indicated by a subscript 1

$$n_e = n_{e0} + n_{e1} \qquad \vec{v_e} = \vec{v_{e0}} + \vec{v_{e1}} \qquad \vec{E} = \vec{E_0} + \vec{E_1}$$

We have n_{e0} and $\vec{v_{e0}}$ as homogeneous and time independent, *i.e.*, $\nabla n_{e0} = \nabla.\,\vec{v_{e0}} = \partial n_{e0}/\partial t = \partial\,\vec{v_{e0}}\,/\partial t = 0$ and $\vec{E_0} = 0$. Then, linearisation of equations (8.18) gives

$$m_e\left[\frac{\partial \vec{v_{e1}}}{\partial t} + (\vec{v_{e0}}\,.\nabla)\,\vec{v_{e1}}\right] = -e\vec{E_1}$$

and

$$\frac{\partial n_{e1}}{\partial t} + n_{e0}\nabla.\,\vec{v_{e1}} + (\vec{v_{e0}}\,.\nabla)n_{e1} = 0 \qquad (8.19)$$

Since the unstable waves are high-frequency plasma oscillations, we must use Poisson equation

$$\epsilon_0\nabla.\,\vec{E} = e(n_i - n_e)$$

where n_i and n_e are densities of ions and electrons, respectively, and an e the electron charge. Linearisation of the Poisson equation gives

$$\epsilon_0\nabla.\,(\vec{E_0} + \vec{E_1}) = e[(n_{i0} + n_{i1}) - (n_{e0} + n_{e1})]$$

$$\epsilon_0\nabla.\,\vec{E_1} = e(n_{i1} - n_{e1}) \qquad (8.20)$$

where we have used $n_{i0} = n_{e0}$, as in the equilibrium, the neutrality is maintained. We assume that the perturbation quantities behave sinusoidally $e^{i(kx-\omega t)}$, then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by $i\vec{k}$. From the equations (8.17), (8.19) and (8.20), we have

$$-i\omega m_i v_{i1} = eE_1 \qquad\qquad v_{i1} = \frac{ie}{m_i\omega}E_1 \qquad (8.21)$$

$$-i\omega n_{i1} + n_0 ikv_{i1} = 0 \qquad n_{i1} = \frac{kn_0}{\omega} v_{i1} \tag{8.22}$$

$$m_e(-i\omega + ikv_0)v_{e1} = -eE_1 \qquad v_{e1} = -\frac{ieE_1}{m_e(\omega - kv_0)} \tag{8.23}$$

$$-i\omega n_{e1} + n_0 ikv_{e1} + ikv_0 n_{e1} = 0 \qquad n_{e1} = \frac{kn_0}{\omega - kv_0} v_{e1} \tag{8.24}$$

$$ik\epsilon_0 E_1 = e(n_{i1} - n_{e1}) \tag{8.25}$$

Using equation (8.21) in (8.22) and (8.23) in (8.24), we get

$$n_{i1} = \frac{kn_0 ie}{\omega m_i \omega} E_1 = \frac{ien_0 k}{m_i \omega^2} E_1 \tag{8.26}$$

and

$$n_{e1} = -\frac{kn_0 ieE_1}{(\omega - kv_0)m_e(\omega - kv_0)} = -\frac{ien_0 k}{m_e(\omega - kv_0)^2} E_1 \tag{8.27}$$

Using equations (8.26) and (8.27) in (8.25), we get

$$ik\epsilon_0 E_1 = e\left[\frac{ien_0 k}{m_i \omega^2} + \frac{ien_0 k}{m_e(\omega - kv_0)^2}\right] E_1$$

Since $E_1 \neq 0$, we get the dispersion relation

$$1 = \frac{ie^2 kn_0}{ik\epsilon_0}\left[\frac{1}{m_i\omega^2} + \frac{1}{m_e(\omega - kv_0)^2}\right]$$

$$1 = \omega_p^2\left[\frac{m_e/m_i}{\omega^2} + \frac{1}{(\omega - kv_0)^2}\right] \tag{8.28}$$

where $\omega_p = \sqrt{n_0 e^2/m_e\epsilon_0}$ is the plasma frequency. This equation (8.28) is a polynomial of fourth-degree for ω. The roots ω_j of the polynomial equation may be real and/or complex numbers. For a real root, sinusoidal behaviour gives

$$e^{i(\vec{k}\cdot\vec{r} - \omega_j t)}$$

showing an oscillatory behaviour. For complex roots which always occur in complex conjugate we write

$$\omega_j = \alpha_j + i\gamma_j$$

where α_j and γ_j are real numbers, and the sinusoidal behaviour gives

$$e^{i(\vec{k}.\vec{r}-\alpha_j t)}\ e^{\gamma_j t}$$

showing an exponentially growing wave for a positive value of γ_j or an exponentially damped wave for a negative value of γ_j. Since the complex roots always appear in a conjugate pair, *i.e.*, we have both negative and positive values of γ_j simultaneously. Hence, for the complex roots, one of the waves is always unstable.

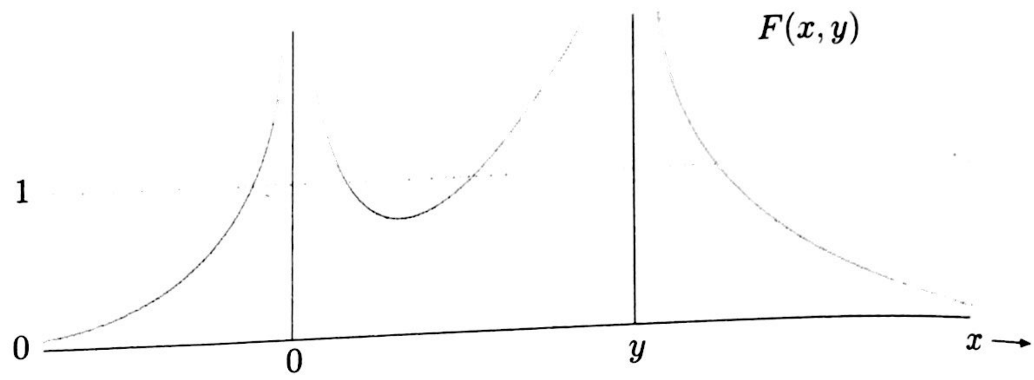

Figure 8.7: Variation of $F(x, y)$ as a function of x for a given value of y. Here, all the four roots are real.

The dispersion relation can be analyzed graphically without solving fourth-order polynomial equation (8.28). Defining

$$x \equiv \frac{\omega}{\omega_p} \qquad \text{and} \qquad y \equiv \frac{kv_0}{\omega_p}$$

equation (8.28) can be written as

$$1 = \frac{m_e/m_i}{x^2} + \frac{1}{(x-y)^2} \equiv F(x, y)$$

The function $F(x, y)$ has singularities at $x = 0$ and at $x = y$. For a given value of y, we can plot the function $F(x, y)$ as a function of x (Figure 8.7). In the figure 8.7, the line $F(x, y) = 1$ intersects the curve at four points, giving the values of x satisfying the dispersion relation. Figure shows that all the four roots of the equation (8.28) are real.

For a smaller value of y, the graph for example is as shown in Figure 8.8, where the line $F(x, y) = 1$ intersects the curve at two points, showing

two real root of the equation (8.28). The other two roots are complex numbers and one of them definitely must correspond to an unstable wave. It shows that for sufficiently small value of kv_0, the plasma is unstable. For a given value of v_0, plasma is unstable to long-wavelength oscillations.

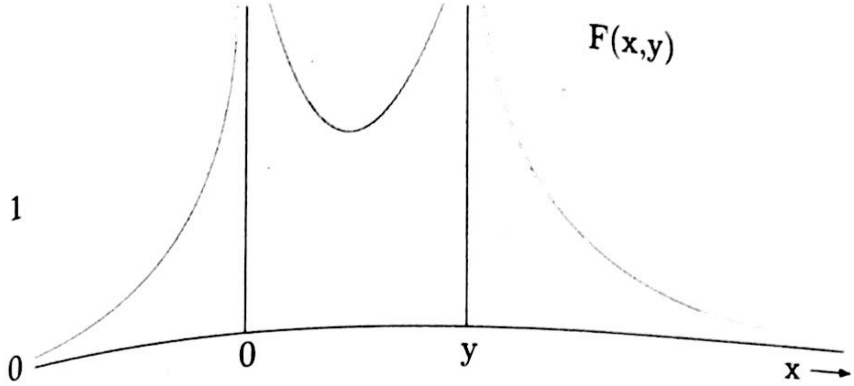

Figure 8.8: Variation of $F(x, y)$ as a function of x for a given value of y. Here, two roots are real.

Since a small value of kv_0 is required for instability, hence for a given value of k, value of v_0 has to be sufficiently small for instability.

8.7 Gravitational instability

A Rayleigh-Taylor instability in a plasma occurs because of the magnetic field acts as a light fluid supporting a heavy fluid (plasma). When the magnetic field lines are curve in shape, the centrifugal force on the plasma due to particle motion along the curved field lines acts as an 'equivalent' gravitational force. Let us consider the simplest case where the plasma surface lies in the yz plane (Figure 8.9). Let the density gradient in the $-x$ direction be ∇n_0 and $\vec{g}$ in the x-direction. Further, we take the case of low-β and cold plasma $T_i = T_e = 0$. Thus, the magnetic field $\vec{B_0}$ is uniform. The equation of motion of the ion is

$$m_i\left[\frac{\partial \vec{v_i}}{\partial t} + (\ \vec{v_i}\ .\nabla)\ \vec{v_i}\ \right] = e\vec{v_i} \times \vec{B_0} + m_i\ \vec{g}$$

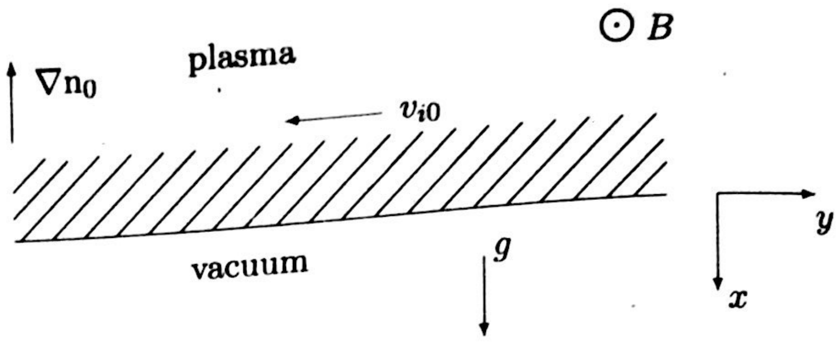

Figure 8.9: A plasma surface subject to a gravitational instability.

For convenience, we drop the subscript i of the ion. In the equilibrium state, we have

$$m_i(\vec{v_0}\cdot\nabla)\vec{v_0} = e\vec{v_0}\times\vec{B_0} + m_i\vec{g} \tag{8.29}$$

If g is constant, v_0 also is constant and therefore $(\vec{v_0}\cdot\nabla)\vec{v_0}$ vanishes. Taking cross product of equation (8.29) with $\vec{B_0}$, we get

$$0 = e(\vec{v_0}\times\vec{B_0})\times\vec{B_0} + m_i\vec{g}\times\vec{B_0}$$

$$0 = e\vec{B_0}(\vec{v_0}\cdot\vec{B_0}) - e\vec{v_0}(\vec{B_0}\cdot\vec{B_0}) + m_i\vec{g}\times\vec{B_0}$$

$$\vec{v}_0 = \frac{m_i}{e}\frac{\vec{g}\times\vec{B_0}}{B_0^2} = -\frac{m_i g}{eB_0}\hat{y} = -\frac{g}{\Omega_c}\hat{y} \tag{8.30}$$

where $\Omega_c(= eB_0/m_i)$ is the ion cyclotron frequency. Because of the opposite charge, electrons would have a drift

$$\vec{v}_{e0} = \frac{g}{\omega_c}\hat{y}$$

This drift of electron can be neglected in the limit $m_e/m_i \to 0$. As we have assumed $T = 0$, there is no diamagnetic drift. Since $\vec{E_0} = 0$, there is no $\vec{E_0}\times\vec{B_0}$ drift also.

If a ripple develops in the interface as a result of random fluctuations, the drift $\vec{v_0}$ causes a ripple to grow. Owing to the drift of the ions, opposite charges are developed on the sides of a ripple and thus an

electric field is developed. The sign of the field changes as we go from a crest to a trough in the perturbation (Figure 8.10).

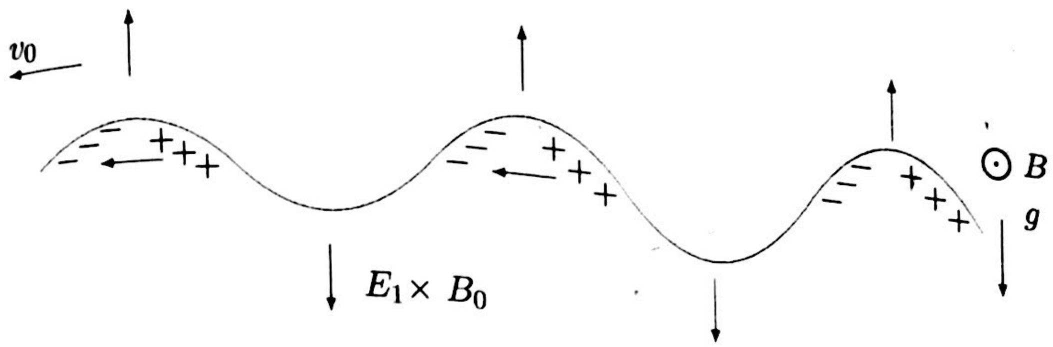

Figure 8.10: Shows a physical mechanism for the gravitational instability.

Figure 8.10 shows that $\vec{E_1} \times \vec{B_0}$ is upward in the regions where the surface has moved upward, and is downward in the regions where the surface has moved downward. The ripple grows as a result of these properly phased $\vec{E_1} \times \vec{B_0}$ drifts.

For finding out the growth rate, let us perform the usual linearisation analysis for the waves propagating in the y direction, $\vec{k} = k\hat{y}$. The equation of motion of the ion is

$$m_i\Big[\frac{\partial \vec{v_i}}{\partial t} + (\vec{v_i} .\nabla)\vec{v_i}\Big] = e\Big[\vec{E} + \vec{v_i} \times \vec{B_0}\Big] + m_i\vec{g}$$

Let us separate the variable parameters into two parts: (i) an equilibrium part, indicated by a subscript 0 and (ii) the perturbation part, indicated by a subscript 1. Then this equation can be written as ($\vec{E_0}$ is zero)

$$m_i\Big[\frac{\partial}{\partial t}(\vec{v_0} + \vec{v_1}) + \Big((\vec{v_0} + \vec{v_1}).\nabla\Big)(\vec{v_0} + \vec{v_1})\Big]$$

$$= e\Big[\vec{E_1} + (\vec{v_0} + \vec{v_1}) \times \vec{B_0}\Big] + m_i\vec{g} \qquad (8.31)$$

Subtracting equation (8.29) from (8.31), we get

$$m_i[\frac{\partial}{\partial t}(\vec{v_0} + \vec{v_1}) + (\vec{v_0} .\nabla)\vec{v_1} + (\vec{v_1} .\nabla)\vec{v_0} + (\vec{v_1} .\nabla)\vec{v_1}] = e[\vec{E_1} + \vec{v_1} \times \vec{B_0}]$$

Though $\vec{g}$ is canceled out, the information about it is still contained in $\vec{v_0}$. On linearisation and treating $\vec{v_0}$ to be homogeneous and independent of time, we get

$$m_i\left[\frac{\partial \vec{v_1}}{\partial t} + (\vec{v_0}\,.\nabla)\,\vec{v_1}\right] = e[\vec{E_1} + \vec{v_1} \times \vec{B_0}] \tag{8.32}$$

We assume that the perturbation quantities behave sinusoidally $e^{i(\vec{k}.\vec{r}-\omega t)}$, then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by $i\vec{k}$. From equation (8.32), we get

$$-m_i i\omega\vec{v_1} + ikm_i v_0\,\vec{v_1} = e(\vec{E_1} + \vec{v_1} \times \vec{B_0})$$

$$m_i(\omega - kv_0)\vec{v_1} = ie(\vec{E_1} + \vec{v_1} \times \vec{B_0})$$

Taking $\vec{v_1} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ and $\vec{E_1} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$, for non-trivial x and y components, we have

$$m_i(\omega - kv_0)v_x = ie(E_x + v_yB_0) \quad \text{and} \quad m_i(\omega - kv_0)v_y = ie(E_y - v_xB_0)$$

Using $\Omega_c = eB_0/m_i$, we have

$$v_x = \frac{ie}{m_i(\omega - kv_0)}\Big(E_x + v_yB_0\Big) = \frac{i\Omega_c}{(\omega - kv_0)}\Big(\frac{E_x}{B_0} + v_y\Big) \tag{8.33}$$

and

$$v_y = \frac{ie}{m_i(\omega - kv_0)}\Big(E_y - v_xB_0\Big) = \frac{i\Omega_c}{(\omega - kv_0)}\Big(\frac{E_y}{B_0} - v_x\Big) \tag{8.34}$$

Using equation (8.34) in (8.33), we have

$$v_x = \frac{i\Omega_c}{(\omega - kv_0)}\Big[\frac{E_x}{B_0} + \frac{i\Omega_c}{(\omega - kv_0)}\Big(\frac{E_y}{B_0} - v_x\Big)\Big]$$

$$= \frac{i\Omega_c}{(\omega - kv_0)}\,\frac{E_x}{B_0} - \frac{\Omega_c^2}{(\omega - kv_0)^2}\,\frac{E_y}{B_0} + \frac{\Omega_c^2}{(\omega - kv_0)^2}\,v_x$$

Thus, we have

$$v_x\Big(1 - \frac{\Omega_c^2}{(\omega - kv_0)^2}\Big) = \frac{\Omega_c}{(\omega - kv_0)B_0}\Big[iE_x - \frac{\Omega_c}{(\omega - kv_0)}E_y\Big]$$

$$= \frac{e}{m_i(\omega - kv_0)}\Big[iE_x - \frac{\Omega_c}{(\omega - kv_0)}E_y\Big]$$

Hence,

$$v_x = \frac{e}{m_i(\omega - kv_0)}\Big[iE_x - \frac{\Omega_c}{(\omega - kv_0)}E_y\Big]\Big(1 - \frac{\Omega_c^2}{(\omega - kv_0)^2}\Big)^{-1}$$

Using equation (8.33) in (8.34), we have

$$v_y = \frac{i\Omega_c}{(\omega - kv_0)}\Big[\frac{E_y}{B_0} - \frac{i\Omega_c}{(\omega - kv_0)}\Big(\frac{E_x}{B_0} + v_y\Big)\Big]$$

$$= \frac{i\Omega_c}{(\omega - kv_0)}\frac{E_y}{B_0} + \frac{\Omega_c^2}{(\omega - kv_0)^2}\frac{E_x}{B_0} + \frac{\Omega_c^2}{(\omega - kv_0)^2}v_y$$

Thus, we have

$$v_y\Big(1 - \frac{\Omega_c^2}{(\omega - kv_0)^2}\Big) = \frac{\Omega_c}{(\omega - kv_0)B_0}\Big[iE_y + \frac{\Omega_c}{(\omega - kv_0)}E_x\Big]$$

$$= \frac{e}{m_i(\omega - kv_0)}\Big[iE_y + \frac{\Omega_c}{(\omega - kv_0)}E_x\Big]$$

Hence,

$$v_y = \frac{e}{m_i(\omega - kv_0)}\Big[iE_y + \frac{\Omega_c}{(\omega - kv_0)}E_x\Big]\Big(1 - \frac{\Omega_c^2}{(\omega - kv_0)^2}\Big)^{-1}$$

For $E_x = 0$ and $\Omega_c^2 >> (\omega - kv_0)^2$, we have

$$v_{ix} = \frac{e}{m_i(\omega - kv_0)}\Big[-\frac{\Omega_c}{(\omega - kv_0)}E_y\Big]\Big[-\frac{(\omega - kv_0)^2}{\Omega_c^2}\Big]$$

$$= \frac{e}{m_i}\frac{E_y}{\Omega_c} = \frac{E_y}{B_0} \tag{8.35}$$

and

$$v_{iy} = \frac{e}{m_i(\omega - kv_0)}iE_y\Big(-\frac{(\omega - kv_0)^2}{\Omega_c^2}\Big)$$

$$= -\frac{ieE_y}{m_i}\frac{(\omega - kv_0)}{\Omega_c}\frac{m_i}{eB_0} = -\frac{i(\omega - kv_{i0})E_y}{\Omega_c B_0} \tag{8.36}$$

Here, we have added the subscript i for ion. The corresponding quantities for the electron are

$$v_{ex} = \frac{E_y}{B_0} \qquad \text{and} \qquad v_{ey} = \frac{i(\omega - kv_{e0})E_y}{\omega_c B_0} \tag{8.37}$$

For the limit $(m_e/m_i) \to 0$, the v_{ey} vanishes. The equation of continuity of an ion is

$$\frac{\partial n_i}{\partial t} + \nabla.(n_i\vec{v_i}) = 0$$

For convenience, we drop the subscript i for ion. Let us separate the variable parameters into two parts: (i) an equilibrium part, indicated by a subscript 0 and (ii) the perturbation part, indicated by a subscript 1.

$$\frac{\partial}{\partial t}(n_0 + n_1) + \nabla.[(n_0 + n_1)(\vec{v_0} + \vec{v_1})] = 0$$

$$\frac{\partial n_1}{\partial t} + \nabla.[n_0\,\vec{v_0} + n_1\,\vec{v_0} + n_0\,\vec{v_1} + n_1\,\vec{v_1}] = 0$$

$$\frac{\partial n_1}{\partial t} + (\vec{v_0}\,.\nabla)n_0 + n_0\nabla.\,\vec{v_0} + (\vec{v_0}\,.\nabla)n_1 + n_1\nabla.\,\vec{v_0}$$

$$+(\vec{v_1}\,.\nabla)n_0 + n_0\nabla.\,\vec{v_1} + (\vec{v_1}\,.\nabla)n_1 + n_1\nabla.\,\vec{v_1} = 0 \tag{8.38}$$

Since $\vec{v_0}$ is perpendicular to ∇n_0 and $\vec{v_0}$ is constant, we have $(\vec{v_0}\,.\nabla)n_0 = n_0\nabla.\,\vec{v_0} = n_1\nabla.\,\vec{v_0} = 0$. Further, the terms $(\vec{v_1}\,.\nabla)n_1$ and $n_1\nabla.\,\vec{v_1}$ are neglected as they are quadratic in perturbation. We assume that the perturbation quantities behave sinusoidally $e^{i(\vec{k}\,.\,\vec{r} - \omega t)}$, then the time derivative $(\partial/\partial t)$ can be replaced by $-i\omega$, and the gradient ∇ by $i\,\vec{k}$. The equation (8.38) gives

$$-i\omega n_1 + ikv_0 n_1 + v_{ix}n_0' + ikn_0 v_{iy} = 0 \tag{8.39}$$

where $n_{i0}' = \partial n_{i0}/\partial x$. Since $\vec{v}_{e0} = 0$ and $v_{ey} = 0$, the corresponding equation for electron is

$$-i\omega n_{e1} + v_{ex}n_{e0}' = 0 \tag{8.40}$$

Here, we have used $n_{i1} = n_{e1}$ as the plasma approximation. This has been possible as the unstable waves are of low frequencies. Equations (8.35), (8.36) and (8.39) give

$$-i(\omega - kv_0)n_1 + \frac{E_y}{B_0}n_0' - ikn_0\frac{i(\omega - kv_0)E_y}{\Omega_c B_0} = 0$$

$$(\omega - kv_0)n_1 + i\frac{E_y}{B_0}n_0' + ikn_0\frac{(\omega - kv_0)}{\Omega_c}\frac{E_y}{B_0} = 0 \tag{8.41}$$

Equations (8.37) and (8.40) give

$$-i\omega n_{e1} + \frac{E_y}{B_0}n_{e0}' = 0 \qquad \frac{E_y}{B_0} = \frac{i\omega n_{e1}}{n_{e0}'} \tag{8.42}$$

Using equation (8.42) in (8.41), we get

$$(\omega - kv_0)n_1 + i\frac{i\omega n_1}{n_0'}n_0' + ikn_0\frac{(\omega - kv_0)}{\Omega_c}\frac{i\omega n_1}{n_0'} = 0$$

$$(\omega - kv_0)n_1 - \left(1 + \frac{kn_0}{n_0'}\frac{(\omega - kv_0)}{\Omega_c}\right)\omega n_1 = 0$$

Since $n_1 \neq 0$, we have

$$\omega - kv_0 - \omega - \frac{kn_0}{n_0'}\frac{\omega(\omega - kv_0)}{\Omega_c} = 0 \qquad \omega(\omega - kv_0) = -\frac{v_0\Omega_c n_0'}{n_0}$$

Using the value of v_0 from equation (8.30), we have

$$\omega(\omega - kv_0) = \frac{g}{\Omega_c}\frac{\Omega_c n_0'}{n_0} = \frac{gn_0'}{n_0} \qquad \omega^2 - kv_0\omega - \frac{gn_0'}{n_0} = 0$$

This quadratic equation for ω has solutions

$$\omega = \frac{kv_0 \pm \sqrt{k^2v_0^2 + 4(gn_0'/n_0)}}{2}$$

Thus, there is instability when ω is complex. That is when

$$k^2v_0^2 + 4(gn_0'/n_0) < 0 \qquad -gn_0'/n_0 > \frac{1}{4}k^2v_0^2$$

It shows that for the gravitational instability, g and n_0'/n_0 must be of the opposite sign. This instability, where $\vec{k} \perp \vec{B}_0$, is sometimes called the *flute instability*.

Real part of ω is $kv_0/2$. As v_0 is the ion velocity, this is a low frequency oscillation, as assumed previously. The factor 1/2 is a consequence of neglecting the electron velocity v_{0e}.

8.7.1 Resistive drift waves

A simple example of a universal instability is the resistive drift waves. In case of the gravitational flute modes, we had $\vec{k}$ perpendicular to $\vec{B}_0$. In case of the drift waves, there is a small but finite component of $\vec{k}$ along $\vec{B}_0$. In this case, the only driving force for the instability is the pressure gradient $KT\nabla n_0$. Here, T is assumed to be constant for simplicity. Let us consider the simplest case where the plasma surface lies in the yz plane (Figure 8.9). Let the density gradient in the $-x$ direction be ∇n_0.

Further, we consider the case of low-β plasma, so that the magnetic field $\vec{B}_0$ is uniform. The equation of motion of the ion is

$$m_i n\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\,.\nabla)\vec{v}\right] = en\vec{v} \times \vec{B}_0 - \nabla p_i$$

In the equilibrium state, we have

$$m_i n_0(\vec{v_0}\,.\nabla)\,\vec{v_0} = en_0\ \vec{v_0} \times \vec{B}_0 - KT_i\nabla n_0 \tag{8.43}$$

As v_0 is constant, the term $(\vec{v_0}\,.\nabla)\vec{v_0}$ vanishes. Taking cross product of equation (8.43) with $\vec{B}_0$, we get

$$0 = en_0(\vec{v_0} \times \vec{B}_0) \times \vec{B}_0 - KT_i\nabla n_0 \times \vec{B}_0$$

$$0 = en_0\ \vec{B}_0\ (\vec{v_0}\ .\ \vec{B}_0) - en_0\ \vec{v_0}\ (\vec{B}_0\ .\ \vec{B}_0) - KT_i\nabla n_0 \times \vec{B}_0$$

$$\vec{v}_{i0} = \vec{v}_{Di} = -\frac{KT_i}{n_0 e}\frac{\nabla n_0 \times \vec{B}_0}{B_0^2} = \frac{KT_i}{eB_0}\frac{n_0'}{n_0}\hat{y}$$

Similarly, for the electron we get

$$\vec{v}_{e0} = \vec{v}_{De} = -\frac{KT_e}{eB_0}\frac{n_0'}{n_0}\ \hat{y}$$

As the drift waves have finite k_z, electrons can flow along $\vec{B}_0$ and establish a thermodynamic equilibrium among themselves. These electrons will obey the Boltzmann relation

$$n_1 = n_0\ \frac{e\phi_1}{KT_e}$$

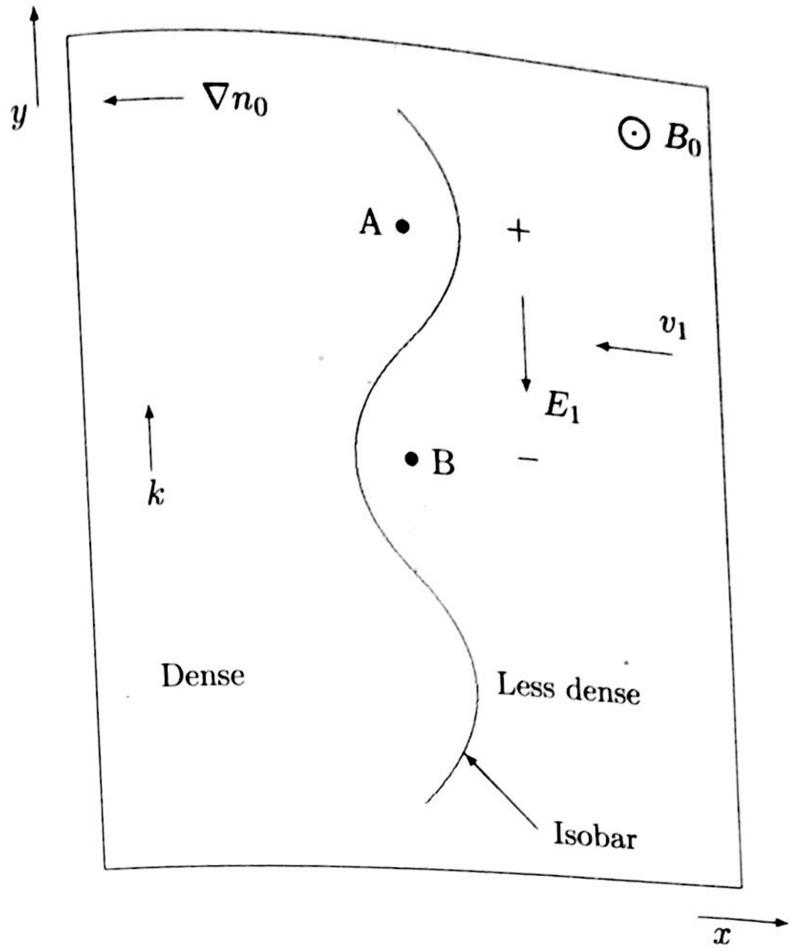

Figure 8.11: Physical mechanism of a drift wave.

In Figure 8.11, at the point A, the density is larger than that in equilibrium. Hence, the deviation of density n_1 is positive, and therefore, ϕ_1 is positive. Similarly, at the point B, the density is smaller than that in equilibrium. Hence, the deviation of density n_1 is negative, and therefore, ϕ_1 is negative. A difference of potential gives rise to an electric field $\vec{E_1}$ between the points A and B, and this electric field causes a drift $\vec{v_1}= (\vec{E_1} \times \vec{B_0})/B_0^2$ in the x direction.

As the wave travels in the y-direction, the values of n_1 and ϕ_1 at any point A oscillates with time. The drift v_1 will also oscillate with time, as it is the v_1 which causes the density to oscillate. Since the gradient ∇n_0 is in the $-x$ direction, the drift v_1 will bring plasma of different density to a given fixed point A. A drift wave though propagates in the y direction, the fluid moves back and forth in the x direction. To be more quantitative, as discussed in the preceding section, the magnitude of v_{1x} is

$$v_{1x} = \frac{E_y}{B_0}$$

Here,

$$E_y = -\frac{\partial \phi_1}{\partial y} = -ik_y\phi_1$$

Therefore,

$$v_{1x} = -\frac{ik_y\phi_1}{B_0}$$

The equation of continuity for a guiding center is

$$\frac{\partial n}{\partial t} + \nabla.(n\vec{v}) = 0$$

$$\frac{\partial(n_0 + n_1)}{\partial t} + \nabla.\Big[(n_0 + n_1)(\vec{v_0} + \vec{v_1})\Big] = 0$$

For the guiding centers, n_0 is homogeneous and independent of time, and $\vec{v_0} = 0$, and thus, we have

$$\frac{\partial n_1}{\partial t} + \nabla.\Big[(n_0 + n_1)\ \vec{v_1}\ \Big] = 0$$

After linearisation, we have

$$\frac{\partial n_1}{\partial t} + n_0\nabla.\ \vec{v_1}\ +(\vec{v_1}\ .\nabla)n_0 = 0$$

Since v_{1x} does not vary with x, the term $n_0\nabla.\ \vec{v_1}$ vanishes, and we have

$$-i\omega n_1 + v_{1x}n_0' = 0$$

where $n_0' = \partial n_0/\partial x$. Using the expressions for n_1 and v_{1x}, we get

$$\omega\frac{e\phi_1}{KT_e}n_0 + \frac{k_y\phi_1}{B_0}n_0' = 0$$

Thus, we have

$$\frac{\omega}{k_y} = -\frac{KT_e}{eB_0}\frac{n_0'}{n_0} = v_{De}$$

showing that these waves travel with the electron diamagnetic drift velocity and are called the *drift waves*.

To see why drift waves are unstable, one must realize that v_{1x} is not exactly E_y/B_0 for the ions. There are corrections due to the polarization

drift and the nonuniform $\vec{E}$ drift. The effect of these drifts is always to make the potential distribution ϕ_1 lag behind the density distribution n_1. This phase shift causes $\vec{v_1}$ to be outward where the plasma has already shifted outward, and vice-versa. When there was no phase shift, n_1 and ϕ_1 would be 90^o out of phase and the drift waves are purely oscillatory.

8.8 Weibel instability

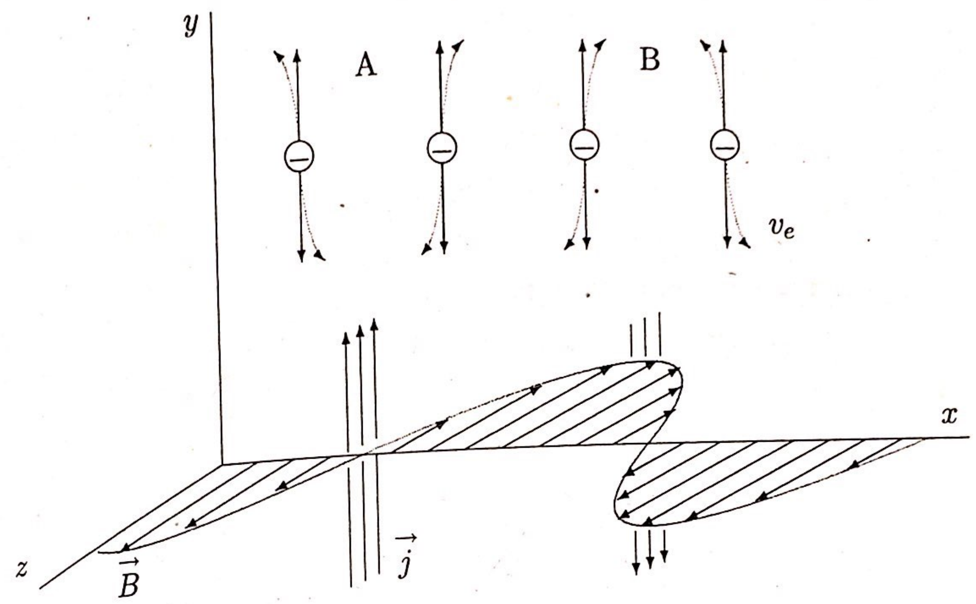

Figure 8.12: Physical mechanism of the Weibel instability.

This is an example of the instability driven by anisotropy of the distribution function where magnetic perturbation is made to grow. This will also serve as an example of electromagnetic instability. Suppose the ions are fix and electrons are hotter in the y direction than that in the x and z directions. Thus, there is preponderance of fast electrons in the $\pm y$ directions (Figure 8.12). However, equal number of electrons flow up and down, so that there is no net current. Suppose a magnetic field $\vec{B}= B\cos(kx)\,\hat{z}$ spontaneously arise from noise. The Lorentz force $-e\vec{v}\times\vec{B}$ then bends the electron motion as shown by dotted curves in the figure with the result that downward moving electrons congregate at A and upward moving at B. The resulting current sheets $\vec{j}= -en_0\,\vec{v}_e$

are phased exactly right to generate a $\vec{B}$ field of the shape assumed, and the perturbation grows. This is a simple treatment of the Weibel instability.

8.9 Problems and questions

1. Discuss about the hydromagnetic equilibrium.

2. Discuss the process of the diffusion of magnetic field into a plasma.

3. Discuss about the two-stream instability in a plasma.

4 Derive condition for a Rayleigh-Taylor instability in a plasma.

5. Write short notes on the following.

    (i) Concept of β

    (ii) Two-stream instability

    (iii) Rayleigh-Taylor instability

    (iv) Resistive drift waves

    (v) Weibel instability

9

Kinetic Theory

The fluid theory used so far is the simplest description of plasma. It is however sufficiently accurate to describe a majority of observed phenomena. But, there are some phenomenon which cannot be described by the fluid theory. For such phenomena, we need to account for the velocity distribution function $f(v)$ for each species. This treatment where velocity distribution function $f(v)$ is used, is known as the *kinetic theory*. In this chapter we shall discuss about the kinetic theory of plasma.

9.1 Meaning of $f(v)$

Difference between the fluid theory and the kinetic theory can be understood with the help of two velocity distributions $f_1(v_x)$ and $f_2(v_x)$ in one dimensional system, shown in Figure 9.1, for example. These two distributions have entirely different behaviours in the kinetic theory, but as along as the areas under the curves are the same, fluid theory provides the same behaviour.

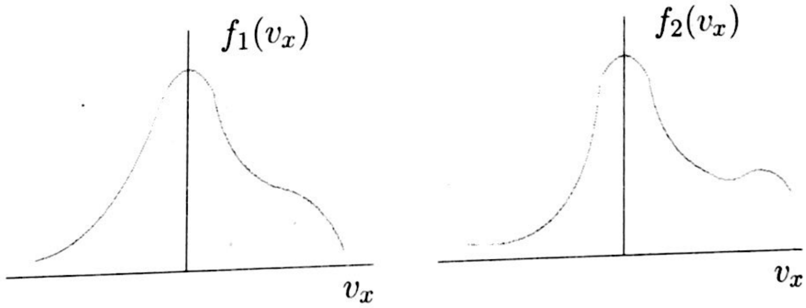

Figure 9.1: Examples of two non-Maxwellian distribution functions in one-dimensional space.

For a distribution function $f(\vec{r}, \vec{v}, t)$, the number of particles per m^3 at position $\vec{r}$ and time t with velocity components in the ranges from

v_x to $v_x + \mathrm{d}v_x$, from v_y to $v_y + \mathrm{d}v_y$, and from v_z to $v_z + \mathrm{d}v_z$ is

$$f(x, y, z, v_x, v_y, v_z, t)\ \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z$$

Here, the position variable $\vec{r}$ and the velocity variable $\vec{v}$ are independent of each other and thus we are dealing in the phase space. The particle density $n(\vec{r}, t)$ is expressed as

$$n(\vec{r}, t) = \int_{-\infty}^{\infty} f(\vec{r}, \vec{v}, t)\ \mathrm{d}\vec{v}$$

where $\mathrm{d}\vec{v}$ represents the volume element in the velocity space. When the function is normalized, it is represented by $\hat{f}(\vec{r}, \vec{v}, t)$ and we have

$$\int_{-\infty}^{\infty} \hat{f}(\vec{r}, \vec{v}, t)\ \mathrm{d}\vec{v} = 1$$

Thus, the function can be expressed as

$$f(\vec{r}, \vec{v}, t) = n(\vec{r}, t)\ \hat{f}(\vec{r}, \vec{v}, t)$$

9.1.1 Maxwellian distribution function

As we know, the normalized Maxwellian distribution function for a gas at temperature T is

$$f_m(v) = \left(\frac{m}{2\pi KT}\right)^{3/2} \exp(-v^2/v_{th}^2)$$

where

$$v \equiv (v_x^2 + v_y^2 + v_z^2)^{1/2} \qquad \text{and} \qquad v_{th} \equiv (2KT/m)^{1/2}$$

It plays an important role for the neutral classical gases. The volume element in the velocity space is $4\pi v^2\ \mathrm{d}v$ (here we assume spherically symmetric distribution in the velocity space) and thus, we have

$$\int_0^{\infty} f_m(v) 4\pi v^2 \mathrm{d}v = \left(\frac{m}{2\pi KT}\right)^{3/2} \int_0^{\infty} \exp(-v^2/v_{th}^2) 4\pi v^2\ \mathrm{d}v = 1$$

It can be verified in the following manner. Let $v = v_{th} x^{1/2}$, and therefore, $\mathrm{d}v = (v_{th}/2) x^{-1/2} \mathrm{d}x$. Then we have

$$I = \left(\frac{m}{2\pi KT}\right)^{3/2} \int_0^{\infty} \exp(-v^2/v_{th}^2) 4\pi v^2\ \mathrm{d}v$$

$$= 4\pi\left(\frac{m}{2\pi KT}\right)^{3/2}\frac{1}{2}\int_0^\infty v_{th}^2 x \ \exp(-x) v_{th} x^{-1/2} \ \mathrm{d}x$$

$$= 2\pi\left(\frac{m}{2\pi KT}\right)^{3/2} v_{th}^3 \int_0^\infty x^{1/2} \ \exp(-x) \ \mathrm{d}x$$

$$= 2\pi\left(\frac{m}{2\pi KT}\right)^{3/2}\left(\frac{2KT}{m}\right)^{3/2}\Gamma(3/2) = 1$$

9.1.2 Average velocity

For the normalized Maxwellian distribution function $f_m(v)$, average velocity of the particles is

$$\langle v\rangle = \int_0^\infty v f_m(v) 4\pi v^2 \mathrm{d}v = 4\pi\left(\frac{m}{2\pi KT}\right)^{3/2}\int_0^\infty v^3 \exp(-v^2/v_{th}^2)\mathrm{d}v$$

Let $v = v_{th}x^{1/2}$, and therefore, $\mathrm{d}v = (v_{th}/2)x^{-1/2}\mathrm{d}x$. Then we have

$$\langle v\rangle = 4\pi\left(\frac{m}{2\pi KT}\right)^{3/2}\frac{1}{2}\int_0^\infty v_{th}^3 x^{3/2} \exp(-x) \ v_{th} x^{-1/2}\mathrm{d}x$$

$$= 2\pi\left(\frac{m}{2\pi KT}\right)^{3/2} v_{th}^4 \int_0^\infty x \ \exp(-x) \ \mathrm{d}x$$

$$= 2\pi\left(\frac{m}{2\pi KT}\right)^{3/2}\left(\frac{2KT}{m}\right)^{2}\Gamma(2) = \left(\frac{8KT}{\pi m}\right)^{1/2}$$

9.1.3 Root mean square velocity

For the normalized Maxwellian distribution function $f_m(v)$, root mean square velocity v_{rms} of particles is

$$v_{rms}^2 = \int_0^\infty v^2 f_m(v) 4\pi v^2 \mathrm{d}v = 4\pi\left(\frac{m}{2\pi KT}\right)^{3/2}\int_0^\infty v^4 \ \exp(-v^2/v_{th}^2)\mathrm{d}v$$

Let $v = v_{th}x^{1/2}$, and therefore, $\mathrm{d}v = (v_{th}/2)x^{-1/2}\mathrm{d}x$. Then we have

$$v_{rms}^2 = 4\pi\left(\frac{m}{2\pi KT}\right)^{3/2}\frac{1}{2}\int_0^\infty v_{th}^4 x^2 \exp(-x) \ v_{th} x^{-1/2}\mathrm{d}x$$

$$= 2\pi\left(\frac{m}{2\pi KT}\right)^{3/2} v_{th}^5 \int_0^\infty x^{3/2} \exp(-x) \ \mathrm{d}x$$

$$= 2\pi\left(\frac{m}{2\pi KT}\right)^{3/2}\left(\frac{2KT}{m}\right)^{5/2}\Gamma(5/2) = \frac{3KT}{m}$$

Thus,

$$v_{rms} = \left(\frac{3KT}{m}\right)^{1/2}$$

9.1.4 Average of a velocity component $|v_x|$

For the normalized Maxwellian distribution function $f_m(v)$, average of x-component of velocity $|v_x|$ of the particles is[1]

$$\langle |v_x| \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |v_x| f_m(\vec{v})\ \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z$$

$$= \left(\frac{m}{2\pi KT}\right)^{3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |v_x|\ \exp\Big[-\ (v_x^2 + v_y^2 + v_z^2)/v_{th}^2\Big]\mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z$$

$$= \left(\frac{m}{2\pi KT}\right)^{3/2}\int_{-\infty}^{\infty} |v_x|\ \exp(-v_x^2/v_{th}^2)\mathrm{d}v_x \int_{-\infty}^{\infty} \exp(-v_y^2/v_{th}^2)\mathrm{d}v_y$$

$$\times \int_{-\infty}^{\infty} \exp(-v_z^2/v_{th}^2)\mathrm{d}v_z$$

Now, one of the integrals is

$$I_1 = \int_{-\infty}^{\infty} \exp(-v_y^2/v_{th}^2)\ \mathrm{d}v_y = 2\int_0^{\infty} \exp(-v_y^2/v_{th}^2)\ \mathrm{d}v_y$$

Here, we used the property that the integrand is symmetric in v_y. Let $v_y = v_{th} x^{1/2}$, and therefore, $\mathrm{d}v_y = (v_{th}/2)x^{-1/2}\mathrm{d}x$. Then we have

$$I_1 = 2\ \frac{1}{2}\int_0^{\infty} \exp(-x)\ v_{th} x^{-1/2}\ \mathrm{d}x = v_{th}\int_0^{\infty} x^{-1/2} \exp(-x)\ \mathrm{d}x$$

$$= \left(\frac{2KT}{m}\right)^{1/2}\Gamma(1/2) = \left(\frac{2\pi KT}{m}\right)^{1/2}$$

Similarly, we have

$$I_2 = \int_{-\infty}^{\infty} \exp(-v_z^2/v_{th}^2)\ \mathrm{d}v_z = \left(\frac{2\pi KT}{m}\right)^{1/2}$$

Now, the remaining integral is

$$I_3 = 2\left(\frac{m}{2\pi KT}\right)^{3/2}\int_0^{\infty} v_x\ \exp(-v_x^2/v_{th}^2)\ \mathrm{d}v_x$$

[1] Volume element in the Cartesian coordinates of velocity space is $\mathrm{d}\vec{v} = \mathrm{d}v_x\ \mathrm{d}v_y\ \mathrm{d}v_z$

Here, we used the property that because of $|v_x|$, the integrand is symmetric in v_x. Let $v_x = v_{th}x^{1/2}$, and therefore, $dv_x = (v_{th}/2)x^{-1/2}dx$. Then we have

$$I_3 = 2\left(\frac{m}{2\pi KT}\right)^{3/2}\frac{1}{2}\int_0^\infty v_{th}x^{1/2}\exp(-x)v_{th}x^{-1/2}dx$$

$$= \left(\frac{m}{2\pi kT}\right)^{3/2} v_{th}^2\int_0^\infty \exp(-x)\, dx$$

$$= \left(\frac{m}{2\pi KT}\right)^{3/2}\left(\frac{2KT}{m}\right)\Gamma(1) = \left(\frac{m}{2\pi^3 KT}\right)^{1/2}$$

On substituting the values of the integrals I_1, I_2, I_3, we have

$$\langle|v_x|\rangle = \left(\frac{2\pi KT}{m}\right)^{1/2}\left(\frac{2\pi KT}{m}\right)^{1/2}\left(\frac{m}{2\pi^3 KT}\right)^{1/2} = \left(\frac{2KT}{\pi m}\right)^{1/2}$$

9.1.5 Average of velocity component v_x

For the normalized Maxwellian distribution function $f_m(v)$, average of x-component of velocity v_x of the particles is

$$\langle v_x\rangle = \int v_x f_m 4\pi v^2 dv = \int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty v_x f_m dv_x dv_y dv_z$$

$$= \int_{-\infty}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty v_x \exp\Big(-(v_x^2+v_y^2+v_z^2)/v_{th}^2\Big)dv_x dv_y dv_z = 0$$

Here, we used the fact that the integrand is an odd function of v_x.

9.1.6 Equations of kinetic theory

The Maxwellian distribution function $f(\vec{r}, \vec{v}, t)$ satisfies the Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v}\,.\nabla f + \frac{\vec{F}}{m}.\frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_c \tag{9.1}$$

where $\vec{F}$ is the force acting on the particle and $(\partial f/\partial t)_c$ the time rate of change of f due to collisions. The ∇ is usual gradient

$$\nabla \equiv \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

and $(\partial/\partial\vec{v})$ or $\nabla_{\vec{v}}$ is

$$\nabla_{\vec{v}} = \frac{\partial}{\partial\vec{v}} \equiv \frac{\partial}{\partial v_x}\hat{x} + \frac{\partial}{\partial v_y}\hat{y} + \frac{\partial}{\partial v_z}\hat{z}$$

Remembering that f is a function of seven independent variables (x, y, z, v_x, v_y, v_z, t), total derivative of f can be expressed as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial f}{\partial v_x}\frac{\mathrm{d}v_x}{\mathrm{d}t} + \frac{\partial f}{\partial v_y}\frac{\mathrm{d}v_y}{\mathrm{d}t} + \frac{\partial f}{\partial v_z}\frac{\mathrm{d}v_z}{\mathrm{d}t} \tag{9.2}$$

Here, $\partial f/\partial t$ is the explicit dependence on time. The next three terms can be written as

$$\frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\hat{x} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{y} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{z}\right).\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right)$$

$$= \vec{v}\,.\nabla f$$

The last three terms in equation (9.2) can be expressed as

$$\frac{\partial f}{\partial v_x}\frac{\mathrm{d}v_x}{\mathrm{d}t} + \frac{\partial f}{\partial v_y}\frac{\mathrm{d}v_y}{\mathrm{d}t} + \frac{\partial f}{\partial v_z}\frac{\mathrm{d}v_z}{\mathrm{d}t} = \left(\frac{\partial f}{\partial v_x}\hat{x} + \frac{\partial f}{\partial v_y}\hat{y} + \frac{\partial f}{\partial v_z}\hat{z}\right).$$

$$\left(\frac{\mathrm{d}v_x}{\mathrm{d}t}\hat{x} + \frac{\mathrm{d}v_y}{\mathrm{d}t}\hat{y} + \frac{\mathrm{d}v_z}{\mathrm{d}t}\hat{z}\right) = \frac{\partial f}{\partial\vec{v}}.\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

Using the Newton's second law

$$\frac{\vec{F}}{m} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

we finally get

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \vec{v}\,.\nabla f + \frac{\vec{F}}{m}.\frac{\partial f}{\partial\vec{v}} \tag{9.3}$$

From equations (9.1) and (9.3), the Boltzmann equation can be expressed as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left(\frac{\partial f}{\partial t}\right)_c$$

The total derivative $\mathrm{d}f/\mathrm{d}t$ can be interpreted as the rate of change as seen in a frame moving with the particles. The Boltzmann equation (9.1) simply says that $\mathrm{d}f/\mathrm{d}t$ is zero unless there are collisions.

9.1.7 Vlasov equation

When the plasma is sufficiently hot, there are no neutral particles and the collisions between the charged particles can be neglected. Now, the force $\vec{F}$ is entirely electromagnetic. Then, equation (9.1) takes the special form

$$\frac{\partial f}{\partial t} + \vec{v} \,.\nabla f + \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right).\frac{\partial f}{\partial \vec{v}} = 0$$

This equation is known as the *Vlasov equation.* Being simpler than other equations, this equation has been most commonly used in the kinetic theory.

9.2 Plasma oscillations and Landau damping

Here, we shall use Vlasov equation for deriving the dispersion relation for electron plasma oscillations. In equilibrium, we assume a uniform plasma with a distribution function $f_0(\vec{v})$, and we take $\vec{B}_0 = \vec{E}_0 = 0$. Let the perturbation in the distribution function $f(\vec{r}, \vec{v}, t)$ be denoted by $f_1(\vec{r}, \vec{v}, t)$ so that

$$f(\vec{r}, \vec{v}, t) = f_0(\vec{v}) + f_1(\vec{r}, \vec{v}, t)$$

The Vlasov equation is

$$\frac{\partial f}{\partial t} + \vec{v} \,.\nabla f + \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right).\frac{\partial f}{\partial \vec{v}} = 0$$

The velocity $\vec{v}$ being an independent variable need not be linearized. For linearisation, let us separate the variable parameters into two parts: (i) an equilibrium part, indicated by a subscript 0 and (ii) the perturbation part, indicated by a subscript 1. Then Vlasov equation for hot plasma can be written as

$$\frac{\partial}{\partial t}(f_0 + f_1) + \vec{v} \,.\nabla(f_0 + f_1) - \frac{e}{m_e}\left[(\vec{E}_0 + \vec{E}_1) + \vec{v} \times \vec{B}_0\right].\frac{\partial}{\partial \vec{v}}(f_0 + f_1) = 0$$

The equilibrium quantities do not depend on time as well as on the space. They however depend on velocity. This equation reduces to

$$\frac{\partial f_1}{\partial t} + \vec{v} \,.\nabla f_1 - \frac{e}{m_e}\,\vec{E}_1 \,.\frac{\partial f_0}{\partial \vec{v}} = 0 \tag{9.4}$$

We assume that the ions being massive are fixed and the electron waves are plane waves in the x direction

$$f_1 \propto \exp[i(kx - \omega t)]$$

equation (9.4) can be expressed as

$$-i\omega f_1 + ikv_x f_1 = \frac{e}{m_e} E_x \frac{\partial f_0}{\partial v_x} \qquad f_1 = \frac{ieE_x \partial f_0/\partial v_x}{m_e(\omega - kv_x)} \tag{9.5}$$

Here, we have taken $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ and $\vec{E_1} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$. Since the electron oscillations are high-frequency plasma oscillations, we must use Poisson equation

$$\epsilon_0 \nabla . \vec{E} = e(n_i - n_e)$$

where n_i and n_e are densities of ions and electrons, respectively, and e the charge of electron. Linearisation of Poisson equation gives

$$\epsilon_0 \nabla .(\vec{E}_0 + \vec{E}_1) = e\Big[n_0 - (n_0 + n_1)\Big] \qquad \epsilon_0 \nabla . \vec{E}_1 = -en_1$$

For the plane waves in the x direction, this equation reduces to

$$ik\epsilon_0 E_x = -en_1 = -e \iiint f_1 \, dv_x dv_y dv_z \tag{9.6}$$

Using equation (9.5) in (9.6), we get

$$ik\epsilon_0 E_x = -e \iiint \frac{ieE_x \partial f_0/\partial v_x}{m_e(\omega - kv_x)} \, dv_x dv_y dv_z$$

Since $E_x \neq 0$, we have

$$1 = -\frac{e^2}{km_e\epsilon_0} \iiint \frac{\partial f_0/\partial v_x}{\omega - kv_x} \, dv_x dv_y dv_z$$

Divide and multiply right side of this equation by n_0 to replace f_0 by a normalized function. Then we have

$$1 = -\frac{\omega_p^2}{k} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial \hat{f}_0/\partial v_x}{\omega - kv_x} \, dv_x \, dv_y \, dv_z \tag{9.7}$$

where $\omega_p = \sqrt{n_0 e^2/m_e\epsilon_0}$ is the plasma frequency. When $\hat{f}_0$ is the normalized Maxwellian distribution function

$$\hat{f}_0 = \Big(\frac{m_e}{2\pi KT_e}\Big)^{3/2} \exp[-(v_x^2 + v_y^2 + v_z^2)/v_{th}^2]$$

where $v_{th} \equiv (2KT_e/m_e)^{1/2}$ is the thermal velocity. The function can be expressed as

$$\hat{f}_0 = f_x(v_x) f_y(v_y) f_z(v_z)$$

where

$$f_x(v_x) = \left(\frac{m_e}{2\pi KT_e}\right)^{1/2} \exp(-v_x^2/v_{th}^2)$$

$$f_y(v_y) = \left(\frac{m_e}{2\pi KT_e}\right)^{1/2} \exp(-v_y^2/v_{th}^2)$$

$$f_z(v_z) = \left(\frac{m_e}{2\pi KT_e}\right)^{1/2} \exp(-v_z^2/v_{th}^2)$$

Now, equation (9.7) can be written as

$$1 = -\frac{\omega_p^2}{k}\int_{-\infty}^{\infty} f_z(v_z)\mathrm{d}v_z \int_{-\infty}^{\infty} f_y(v_y)\mathrm{d}v_y \int_{-\infty}^{\infty} \frac{\partial f_x(v_x)/\partial v_x}{\omega - kv_x}\,\mathrm{d}v_x \tag{9.8}$$

We have

$$\int_{-\infty}^{\infty} f_y(v_y)\mathrm{d}v_y = \left(\frac{m_e}{2\pi KT}\right)^{1/2} \int_{-\infty}^{\infty} \exp(-v_y^2/v_{th}^2)\,\mathrm{d}v_y = 1$$

$$\int_{-\infty}^{\infty} f_z(v_z)\mathrm{d}v_z = \left(\frac{m_e}{2\pi KT}\right)^{1/2} \int_{-\infty}^{\infty} \exp(-v_z^2/v_{th}^2)\mathrm{d}v_z = 1$$

Thus, equation (9.8) reduces to

$$1 = \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{\partial f_x(v_x)/\partial v_x}{v_x - (\omega/k)}\,\mathrm{d}v_x$$

Since we are now dealing with one-dimensional case, for convenience, we can drop the suffix x, and can write

$$1 = \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} \frac{\partial f(v)/\partial v}{v - (\omega/k)}\,\mathrm{d}v \tag{9.9}$$

This equation has a pole at $v = (\omega/k)$ and therefore it requires a special treatment. This equation was solved properly by Landau where he prescribed a contour as a straight line (Figure 9.2) along the $\mathrm{Re}(v)$ with a

small semicircle around the pole. In going around the pole, one obtains $2\pi i$ times half the residue there. Then equation (9.9) gives

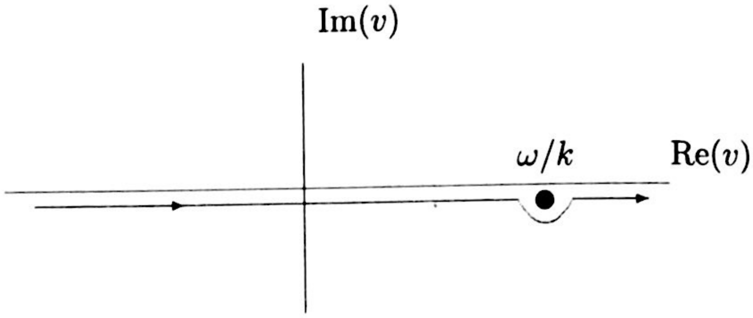

Figure 9.2: Integration contour in the complex v plane. There is a pole on the real axis at $v = \omega/k$.

$$1 = \frac{\omega_p^2}{k^2}\left[P\int_{-\infty}^{\infty}\frac{\partial f(v)/\partial v}{v-(\omega/k)}\,\mathrm{d}v + i\pi\frac{\partial f(v)}{\partial v}\bigg|_{v=(\omega/k)}\right] \tag{9.10}$$

where P stands for the Cauchy principle value, which is generally taken equal to one. This is the dispersion relation. In order to evaluate this, we integrate along the real v axis but stop just before the pole. If the phase velocity $v_\phi = \omega/k$ is sufficiently high, there will not be much contribution from the neglected part of the contour, as both f and $\partial f/\partial v$ are very small there. The integral in (9.10) can be evaluated as

$$\int_{-\infty}^{\infty}\frac{\partial f(v)}{\partial v}\frac{\mathrm{d}v}{v-v_\phi} = \left[\frac{f}{v-v_\phi}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{-f\,\mathrm{d}v}{(v-v_\phi)^2} = \int_{-\infty}^{\infty}\frac{f\,\mathrm{d}v}{(v-v_\phi)^2}$$

The value of f tends to zero at $v = \pm\infty$. This is just an average of $(v-v_\phi)^{-2}$ over the distribution. Since $v_\phi >> v$, we can expand $(v-v_\phi)^{-2}$ as

$$(v-v_\phi)^{-2} = v_\phi^{-2}\left(1-\frac{v}{v_\phi}\right)^{-2} = v_\phi^{-2}\left(1+\frac{2v}{v_\phi}+\frac{3v^2}{v_\phi^2}+\frac{4v^3}{v_\phi^3}+\ldots\right)$$

On taking the average, the odd terms vanish and we

$$\left\langle(v-v_\phi)^{-2}\right\rangle \approx v_\phi^{-2}\left(1+\frac{3\langle v^2\rangle}{v_\phi^2}\right)$$

Remembering the v is v_x, we have

$$\frac{1}{2}\,m_e\langle v^2\rangle = \frac{1}{2}\,KT_e$$

we have

$$\left\langle (v - v_\phi)^{-2} \right\rangle \approx v_\phi^{-2}\left(1 + \frac{3KT_e}{m_e v_\phi^2}\right) = \frac{k^2}{\omega^2}\left(1 + \frac{k^2}{\omega^2}\frac{3KT_e}{m_e}\right)$$

Hence, the real part of equation (9.10) is

$$1 = \frac{\omega_p^2}{k^2}\frac{k^2}{\omega^2}\left(1 + 3\,\frac{k^2}{\omega^2}\frac{KT_e}{m_e}\right) \qquad \omega^2 = \omega_p^2 + \frac{\omega_p^2}{\omega^2}\frac{3KT_e}{m_e}\,k^2$$

When the thermal correction is small, we may replace ω^2 by ω_p^2 in the second term and obtain

$$\omega^2 = \omega_p^2 + \frac{3KT_e}{m_e}\,k^2$$

In evaluating small terms, we can neglect the thermal corrections and it would not affect the result significantly. Equation (9.9) can therefore be written as

$$1 = \frac{\omega_p^2}{\omega^2}\left[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\right] + \frac{i\pi\omega_p^2}{k^2}\left.\frac{\partial f(v)}{\partial v}\right|_{v=v_\phi}$$

$$1 - \frac{i\pi\omega_p^2}{k^2}\left.\frac{\partial f(v)}{\partial v}\right|_{v=v_\phi} = \frac{\omega_p^2}{\omega^2}\left[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\right]$$

$$\omega^2\left(1 - \frac{i\pi\omega_p^2}{k^2}\left.\frac{\partial f(v)}{\partial v}\right|_{v=v_\phi}\right) = \omega_p^2\left[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\right]$$

$$\omega = \omega_p\left[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\right]^{1/2}\left(1 - \frac{i\pi\omega_p^2}{k^2}\left.\frac{\partial f(v)}{\partial v}\right|_{v=v_\phi}\right)^{-1/2}$$

$$\omega = \omega_p\left[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\right]^{1/2}\left(1 + \frac{i\pi\omega_p^2}{2k^2}\left.\frac{\partial f(v)}{\partial v}\right|_{v=v_\phi}\right) \tag{9.11}$$

Remember that $f(v)$ here is one-dimensional Maxwellian distribution function

$$f(v) = \left(\frac{m_e}{2\pi KT}\right)^{1/2}\exp\left(-\frac{v^2}{v_{th}^2}\right)$$

$$\frac{\partial f(v)}{\partial v} = -\frac{2v}{v_{th}^3\sqrt{\pi}}\,\exp\left(-\frac{v^2}{v_{th}^2}\right)$$

$$\left.\frac{\partial f(v)}{\partial v}\right|_{v=v_\phi} = -\frac{2v_\phi}{v_{th}^3\sqrt{\pi}}\,\exp\left(-\frac{v_\phi^2}{v_{th}^2}\right)$$

Thus, equation (9.11) gives

$$\omega = \omega_p\Big[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\Big]^{1/2}\Big(1 - \frac{i\pi\omega_p^2 v_\phi}{k^2 v_{th}^3\sqrt{\pi}}\exp\Big(-\frac{v_\phi^2}{v_{th}^2}\Big)\Big)$$

Then the damping is

$$\begin{aligned}
\mathrm{Im}(\omega) &= -\frac{\sqrt{\pi}\omega_p^3}{k^2 v_{th}^3}\Big[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\Big]^{1/2} v_\phi \exp\Big(-\frac{v_\phi^2}{v_{th}^2}\Big) \\
&= -\frac{\sqrt{\pi}\omega_p^3}{k^2 v_{th}^3}\Big[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\Big]^{1/2}\frac{\omega}{k}\exp\Big(-\frac{\omega^2}{k^2 v_{th}^2}\Big) \\
&= -\frac{\sqrt{\pi}\omega_p^3}{k^3 v_{th}^3}\Big[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\Big]\exp\Big[-\frac{\omega_p^2 + (3KT_e/m_e)k^2}{k^2 v_{th}^2}\Big] \\
&= -\sqrt{\pi}\Big(\frac{\omega_p}{kv_{th}}\Big)^3\Big[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\Big]\exp\Big(-\frac{\omega_p^2}{k^2 v_{th}^2}\Big)\exp\Big(-\frac{3}{2}\Big) \\
&= -0.22\sqrt{\pi}\Big(\frac{\omega_p}{kv_{th}}\Big)^3\Big[1 + \frac{3KT_e k^2}{m_e\omega_p^2}\Big]\exp\Big(-\frac{1}{2k^2\lambda_D^2}\Big)
\end{aligned}$$

where λ_D is the Debye length.[2] As $\mathrm{Im}(\omega)$ is negative, there is a collisionless damping of plasma waves. This is known as the *Landau damping*. When $k\lambda_D$ is small, the damping is very small, but it becomes important when $k\lambda_D$ is of the order of unity.

9.3 Ion Landau damping

Here, we shall use Vlasov equation to derive the dispersion relation for ion oscillations. In equilibrium, we assume a uniform plasma with a distribution function $f_0(\vec{v})$, and we take $\vec{B}_0 = \vec{E}_0 = 0$. Let the perturbation in the distribution function $f(\vec{r}, \vec{v}, t)$ be denoted by $f_1(\vec{r}, \vec{v}, t)$ so that

$$f(\vec{r}, \vec{v}, t) = f_0(\vec{v}) + f_1(\vec{r}, \vec{v}, t)$$

[2] We have

$$\frac{\omega_p^2}{v_{th}^2} = \frac{n_0 e^2}{m_e\epsilon_0}\,\frac{m_e}{2kT_e} = \frac{1}{2}\left(\frac{n_0 e^2}{\epsilon_0 kT_e}\right) = \frac{1}{2\lambda_D^2}$$

The Vlasov equation is

$$\frac{\partial f}{\partial t}+\vec{v}\,.\nabla f+\frac{q}{m}\Big(\vec{E}+\vec{v}\times\vec{B}\Big).\frac{\partial f}{\partial \vec{v}}=0$$

Here, $\vec{v}$ being an independent variable need not to be linearized. For linearisation, let us separate the variable parameters into two parts: (i) an equilibrium part, indicated by a subscript 0 and (ii) the perturbation part, indicated by a subscript 1. Then Vlasov equation for hot plasma can be written as

$$\frac{\partial}{\partial t}(f_0+f_1)+\vec{v}\,.\nabla(f_0+f_1)+\frac{e}{m_i}\Big((\vec{E}_0+\vec{E}_1)+\vec{v}\times\vec{B}_0\Big).\frac{\partial}{\partial \vec{v}}(f_0+f_1)=0$$

The equilibrium quantities do not depend on time as well as on the space. They however depend on velocity. This equation reduces to

$$\frac{\partial f_1}{\partial t}+\vec{v}\,.\nabla f_1+\frac{e}{m_i}\,\vec{E}_1\,.\frac{\partial f_0}{\partial \vec{v}}=0 \qquad (9.12)$$

We assume that ion waves are plane waves in the x direction

$$f_1 \propto \exp[i(kx-\omega t)]$$

equation (9.12) can be expressed as

$$-i\omega f_1+ikv_x f_1=-\frac{e}{m_i}E_x\frac{\partial f_0}{\partial v_x} \qquad f_1=-\frac{ieE_x\partial f_0/\partial v_x}{m_i(\omega-kv_x)} \qquad (9.13)$$

Motion of electrons parallel to $\vec{B}$ is the same as in absence of $\vec{B}$. For the electric field $\vec{E}=-\nabla\phi=-\nabla(\phi_0+\phi_1)=-\nabla\phi_1$, the density of massless electrons is[3]

$$n_e=n=n_0\exp\Big(\frac{e\phi_1}{KT_e}\Big)$$

$$=n_0\Big(1+\frac{e\phi_1}{KT_e}+\dots\Big)=n_0\Big(1+\frac{e\phi_1}{KT_e}\Big)$$

Thus,

$$n-n_0=n_0\,\frac{e\phi_1}{KT_e} \qquad n_1=n_0\,\frac{e\phi_1}{KT_e} \qquad (9.14)$$

[3]Mass of an electron may be considered as negligible in comparison to that of proton

The relation $\vec{E} = -\nabla\phi_1$ gives

$$E_x = -ik\phi_1 \tag{9.15}$$

From equations (9.14) and (9.15), we have

$$E_x = -ik\frac{KT_e}{n_0 e}\, n_1 = -ik\,\frac{KT_e}{n_0 e}\iiint f_1 \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z \tag{9.16}$$

Using equation (9.13) in (9.16), we get

$$E_x = -ik\frac{KT_e}{n_0 e}\iiint -\frac{ieE_x \partial f_0/\partial v_x}{m_i(\omega - kv_x)}\mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z$$

Since $E_x \neq 0$, we have

$$1 = -\frac{kKT_e}{n_0 m_i}\iiint \frac{\partial f_0/\partial v_x}{\omega - kv_x}\,\mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z$$

Division and multiplication by n_0 on the right side of this equation replaces f_0 by a normalized function and we have

$$1 = -\frac{kKT_e}{m_i}\int_{-\infty}^{\infty}\mathrm{d}v_z \int_{-\infty}^{\infty}\mathrm{d}v_y \int_{-\infty}^{\infty}\frac{\partial \hat{f}_0/\partial v_x}{\omega - kv_x}\,\mathrm{d}v_x \tag{9.17}$$

When $\hat{f}_0$ is the Maxwellian distribution function

$$\hat{f}_0 = \left(\frac{m_i}{2\pi KT_i}\right)^{3/2}\exp\{-(v_x^2 + v_y^2 + v_z^2)/v_{th}^2\}$$

where $v_{th} \equiv (2KT_i/m_i)^{1/2}$ is the thermal velocity, it can be expressed as

$$\hat{f}_0 = f_x(v_x) f_y(v_y) f_z(v_z)$$

where

$$f_x(v_x) = \left(\frac{m_i}{2\pi KT_i}\right)^{1/2}\exp(-v_x^2/v_{th}^2)$$

$$f_y(v_y) = \left(\frac{m_i}{2\pi KT_i}\right)^{1/2}\exp(-v_y^2/v_{th}^2)$$

$$f_z(v_z) = \left(\frac{m_i}{2\pi KT_i}\right)^{1/2}\exp(-v_z^2/v_{th}^2)$$

Now, equation (9.17) can be written as

$$1 = -\frac{kKT_e}{m_i}\int_{-\infty}^{\infty} f_z(v_z)\,\mathrm{d}v_z \int_{-\infty}^{\infty} f_y(v_y)\,\mathrm{d}v_y \int_{-\infty}^{\infty}\frac{\partial f_x(v_x)/\partial v_x}{\omega - kv_x}\,\mathrm{d}v_x \tag{9.18}$$

We have

$$\int_{-\infty}^{\infty} f_y(v_y)\,dv_y = \left(\frac{m_i}{2\pi KT_i}\right)^{1/2}\int_{-\infty}^{\infty} \exp(-v_y^2/v_{th}^2)dv_y = 1$$

$$\int_{-\infty}^{\infty} f_z(v_z)\,dv_z = \left(\frac{m_i}{2\pi KT_i}\right)^{1/2}\int_{-\infty}^{\infty} \exp(-v_z^2/v_{th}^2)dv_z = 1$$

Thus, equation (9.18) reduces to

$$1 = \frac{KT_e}{m_i}\int_{-\infty}^{\infty} \frac{\partial f_x(v_x)/\partial v_x}{v_x - (\omega/k)}\,dv_x$$

Since we are now dealing with one-dimensional case, for convenience, we can drop the suffix x, and can write

$$1 = \frac{KT_e}{m_i}\int_{-\infty}^{\infty} \frac{\partial f(v)/\partial v}{v - (\omega/k)}\,dv \tag{9.19}$$

This equation has a pole at $v = (\omega/k)$ and therefore needs a special treatment. This equation was solved properly by Landau where he prescribed a contour as a straight line (Figure 2) along the Re(v) with a small semicircle around the pole. In going around the pole, one obtains $2\pi i$ times half the residue there. Then equation (9.19) gives

$$1 = \frac{KT_e}{m_i}\left[P\int_{-\infty}^{\infty} \frac{\partial f(v)/\partial v}{v - (\omega/k)}\,dv + i\pi\frac{\partial f(v)}{\partial v}\Big|_{v=(\omega/k)}\right] \tag{9.20}$$

where P stands for the Cauchy principle value, which is generally taken equal to one. This is the dispersion relation. In order to evaluate this, we integrate along the real v axis but stop just before the pole. If the phase velocity $v_\phi = \omega/k$ is sufficiently high, there will not be much contribution from the neglected part of the contour, as both f and $\partial f/\partial v$ are very small there. The integral in (9.20) can be evaluated as

$$\int_{-\infty}^{\infty} \frac{\partial f(v)}{\partial v}\frac{dv}{v - v_\phi} = \left[\frac{f}{v - v_\phi}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{-f\,dv}{(v - v_\phi)^2} = \int_{-\infty}^{\infty}\frac{f\,dv}{(v - v_\phi)^2}$$

The value of f tends to zero at $v = \pm\infty$. This is just an average of $(v - v_\phi)^{-2}$ over the distribution. Since $v_\phi >> v$, we can expand $(v - v_\phi)^{-2}$ as

$$(v - v_\phi)^{-2} = v_\phi^{-2}\left(1 - \frac{v}{v_\phi}\right)^{-2} = v_\phi^{-2}\left(1 + \frac{2v}{v_\phi} + \frac{3v^2}{v_\phi^2} + \frac{4v^3}{v_\phi^3} + \dots\right)$$

On taking the average, the odd terms vanish and we have

$$\left\langle (v - v_\phi)^{-2} \right\rangle \approx v_\phi^{-2}\left(1 + \frac{3\langle v^2 \rangle}{v_\phi^2}\right)$$

Remembering the v is v_x, we have

$$\frac{1}{2}\, m_i \langle v^2 \rangle = \frac{1}{2}\, KT_i$$

Therefore,

$$\left\langle (v - v_\phi)^{-2} \right\rangle \approx v_\phi^{-2}\left(1 + \frac{3KT_i}{m_i v_\phi^2}\right) = \frac{k^2}{\omega^2}\left(1 + \frac{k^2 3KT_i}{\omega^2 m_i}\right)$$

Hence, the real part of equation (9.20) is

$$1 = \frac{KT_e k^2}{m_i \omega^2}\left(1 + \frac{3k^2 KT_i}{\omega^2 m_i}\right) \qquad \frac{\omega^2}{k^2} = \frac{KT_e}{m_i} + \frac{3KT_e k^2 KT_i}{m_i \omega^2 m_i}$$

When the thermal correction is small, we may replace ω^2/k^2 by KT_e/m_i in the second term and obtain

$$\frac{\omega^2}{k^2} = \frac{KT_e + 3KT_i}{m_i} \qquad \frac{\omega}{k} = \left(\frac{KT_e + 3KT_i}{m_i}\right)^{1/2}$$

In evaluating small terms, we can neglect the thermal corrections and it would not affect the result significantly. The equation (9.20) can therefore be written as

$$1 = \frac{k^2(KT_e + 3KT_i)}{\omega^2 m_i} + \frac{i\pi KT_e}{m_i}\,\frac{\partial f(v)}{\partial v}\bigg|_{v=v_\phi}$$

$$1 - \frac{i\pi KT_e}{m_i}\frac{\partial f(v)}{\partial v}\bigg|_{v=v_\phi} = \frac{k^2(KT_e + 3KT_i)}{m_i \omega^2}$$

$$\omega^2\left(1 - \frac{i\pi KT_e}{m_i}\frac{\partial f(v)}{\partial v}\bigg|_{v=v_\phi}\right) = \frac{k^2(KT_e + 3KT_i)}{m_i}$$

$$\omega = k\left(\frac{KT_e + 3KT_i}{m_i}\right)^{1/2}\left(1 - \frac{i\pi KT_e}{m_i}\,\frac{\partial f(v)}{\partial v}\bigg|_{v=v_\phi}\right)^{-1/2}$$

$$= k\left(\frac{KT_e + 3KT_i}{m_i}\right)^{1/2}\left(1 + \frac{i\pi KT_e}{2m_i}\frac{\partial f(v)}{\partial v}\bigg|_{v=v_\phi}\right) \qquad (9.21)$$

Here, $f(v)$ is one-dimensional Maxwellian distribution function

$$f(v) = \left(\frac{m_i}{2\pi KT_i}\right)^{1/2} \exp\left(-\frac{v^2}{v_{th}^2}\right)$$

$$\frac{\partial f(v)}{\partial v} = \frac{1}{v_{th}\sqrt{\pi}}\left(-\frac{2v}{v_{th}^2}\right)\exp\left(-\frac{v^2}{v_{th}^2}\right)$$

$$= -\frac{2v}{v_{th}^3\sqrt{\pi}}\exp\left(-\frac{v^2}{v_{th}^2}\right)$$

$$\left.\frac{\partial f(v)}{\partial v}\right|_{v=v_\phi} = -\frac{2v_\phi}{v_{th}^3\sqrt{\pi}}\exp\left(-\frac{v_\phi^2}{v_{th}^2}\right)$$

Thus, equation (9.21) gives

$$\omega = k\left(\frac{KT_e + 3KT_i}{m_i}\right)^{1/2}\left(1 - \frac{i\pi KT_e}{2m_i}\frac{2v_\phi}{v_{th}^3\sqrt{\pi}}\exp\left(-\frac{v_\phi^2}{v_{th}^2}\right)\right)$$

Then the damping is

$$\mathrm{Im}(\omega) = -\frac{\sqrt{\pi}k}{v_{th}^3}\left(\frac{KT_e + 3KT_i}{m_i}\right)^{1/2}\frac{KT_e}{m_i}v_\phi \exp\left(-\frac{v_\phi^2}{v_{th}^2}\right)$$

$$= -\frac{\sqrt{\pi}k}{v_{th}^3}\left(\frac{KT_e + 3KT_i}{m_i}\right)\frac{KT_e}{m_i}\exp\left(-\frac{KT_e + 3KT_i}{2KT_i}\right)$$

$$= -\frac{\sqrt{\pi}k}{v_{th}}\left(\frac{KT_e + 3KT_i}{2KT_i}\right)\frac{KT_e}{m_i}\exp\left(-\frac{T_e}{2T_i}\right)\exp(-3/2)$$

$$= -0.22\frac{\sqrt{\pi}k}{v_{th}}\left(\frac{T_e + 3T_i}{2T_i}\right)\frac{KT_e}{m_i}\exp\left(-\frac{T_e}{2T_i}\right)$$

As $\mathrm{Im}(\omega)$ is negative, there is a collision-less damping of ion waves. This is known as the *Landau ion damping*. When $T_e > T_i$, Landau ion damping is very small. But, when $T_e \le T_i$, Landau ion damping is quite significant.

9.3.1 Kinetic effects in a magnetic field

When the magnetic field $\vec{B}_0$ or the oscillating magnetic field $\vec{B}_1$ is finite. The Vlasov equation is

$$\frac{\partial f}{\partial t} + \vec{v}\,.\nabla f + \frac{q}{m}\left(\vec{E} + \vec{v}\times\vec{B}\right).\frac{\partial f}{\partial \vec{v}} = 0$$

For linearisation, let us separate the variable parameters into two parts: (i) an equilibrium part, indicated by a subscript 0 and (ii) the perturbation part, indicated by a subscript 1. With $\vec{E_0} = 0$, Vlasov equation for hot plasma can be written as

$$\frac{\partial}{\partial t}(f_0 + f_1) + \vec{v} \cdot \nabla (f_0 + f_1) + \frac{q}{m}\left[\vec{E_1} + \vec{v} \times (\vec{B_0} + \vec{B_1})\right] \cdot \frac{\partial}{\partial \vec{v}}(f_0 + f_1) = 0$$

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \nabla f_1 + \frac{q}{m}\left(\vec{v} \times \vec{B_0}\right) \cdot \frac{\partial f_1}{\partial \vec{v}} = -\frac{q}{m}\left(\vec{E_1} + \vec{v} \times \vec{B_1}\right) \cdot \frac{\partial f_0}{\partial \vec{v}}$$

The particles moving along $\vec{B_0}$ still show Landau damping when $\omega/k \approx v_{th}$. For the velocity component $v_\perp$ perpendicular to $\vec{B_0}$, two new kinetic effects appear: (i) cyclotron harmonics, leading to oscillations, known as the *Bernstein waves* and (ii) cyclotron damping .

9.3.2 Cyclotron harmonics (Bernstein waves)

Harmonics of the cyclotron frequency are generated when the particles circular Larmor orbits are distorted by the wave fields $\vec{E_1}$ and $\vec{B_1}$. In order to understand the process of production of harmonics, let us consider motion of particles in an electric field

$$\vec{E} = E_x \, e^{i(kx-\omega t)} \, \hat{i}$$

Equation of motion of the particle is

$$m \frac{d\vec{v}}{dt} = q[\vec{E} + \vec{v} \times \vec{B}] \tag{9.22}$$

Here, we use the Cartesian coordinates. Let us take $\vec{B} = B\hat{k}$, where B is magnitude of the field. For the velocity

$$\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} \tag{9.23}$$

we have

$$\vec{v} \times \vec{B} = v_y B\hat{i} - v_x B\hat{j} \tag{9.24}$$

Using equations (9.23) and (9.24) in (9.22) and equating the coefficients of $\hat{i}$ and $\hat{j}$ on the two sides of the resulting equation, we get

$$m \, \dot{v_x} = qE_x e^{i(kx-\omega t)} + qBv_y \tag{9.25}$$

$$m\,\dot{v}_y = -qBv_x \tag{9.26}$$

where a dot on a quantity represents its differentiation with respect to time t. Integration of equation (9.26) gives

$$v_y = -\frac{qB}{m}x \tag{9.27}$$

where constant of integration is taken zero. Using equation (9.27) in (9.25) we get

$$\dot{v}_x = \frac{q}{m}E_x\,\mathrm{e}^{i(kx-\omega t)} - \left(\frac{qB}{m}\right)^2 x \qquad \ddot{x} + \omega_c^2 x = \frac{q}{m}E_x\,\mathrm{e}^{i(kx-\omega t)}$$

where $\omega_c = |q|B/m$ is the cyclotron frequency. As an approximation, let us take x as the undisturbed orbit $x = r_L \sin(\omega_c t)$ so that

$$\ddot{x} + \omega_c^2 x = \frac{q}{m}E_x\,\mathrm{e}^{i(kr_L \sin\omega_c t - \omega t)} \tag{9.28}$$

The generating function for the Bessel function $J_n(z)$ is

$$\mathrm{e}^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$$

Putting $z = kr_L$ and $t = \exp(i\omega_c t)$, we get

$$\mathrm{e}^{ikr_L \sin\omega_c t} = \sum_{n=-\infty}^{\infty} J_n(kr_L)\,\mathrm{e}^{in\omega_c t} \tag{9.29}$$

Using equation (9.29) in (9.28), we have

$$\ddot{x} + \omega_c^2 x = \frac{q}{m}E_x \sum_{n=-\infty}^{\infty} J_n(kr_L)\,\mathrm{e}^{-i(\omega - n\omega_c)t}$$

This equation has the following solution which can be verified by direct substitution

$$x = \frac{q}{m}E_x \sum_{n=-\infty}^{\infty} \frac{J_n(kr_L)\,\mathrm{e}^{-i(\omega-n\omega_c)t}}{\omega_c^2 - (\omega - n\omega_c)^2}$$

This obviously shows that the motion has frequency components differing from the driving frequency ω by multiples of ω_c and the amplitude of these components are proportional to $J_n(kr_L)/[\omega_c^2 - (\omega - n\omega_c)^2]$. There is singularity when $\omega - n\omega_c = \pm\omega_c$ or $\omega = (n \pm 1)\omega_c$, where $n = 0, \pm1, \pm2, \ldots$. This singularity occurs when the field $\vec{E}\,(x,t)$ resonates with any of the harmonic of ω_c.

The electrostatic waves propagating at right angles to the magnetic field $\vec{B}_0$ at the frequencies which are harmonics of the cyclotron frequency are known as *Bernstein waves*.

9.3.3 Cyclotron damping

When a particle moving along $\vec{B}_0$ in a wave with finite k_z, the particle sees a Doppler shifted frequency $\omega - k_z v_z$. When this frequency is equal to $\pm n\omega_c$, it is subject to continuous acceleration by the electric field $\vec{E}_\perp$ of the wave. Those particles with the *right* phase relative to $\vec{E}_\perp$ will gain energy whereas those with the *wrong* phase will lose energy. Since the change in energy is the force multiplied by the distance, the accelerated particles gain more energy per unit time than those decelerated particles lose. There is a net gain of energy by the particles on the average, at the expense of the wave energy, and therefore, the wave is damped. This mechanism of damping is different from the Landau damping as the energy gained is in the direction perpendicular to $\vec{B}_0$ and thus perpendicular to the velocity component that brings the particle into resonance.

9.4 Problems and questions

1. For a Maxwellian distribution function, obtain average velocity, root mean square velocity, average of velocity component $|v_x|$, and average of velocity component v_x.

2. Discuss about the Boltzmann equation and derive fluid equations from it.

3. Discuss about the Landau damping for electron waves.

4. Discuss about the Landau damping for ion waves.

5. Write short notes on the following

   (i) Vlasov equation

   (ii) Landau damping

   (iii) Bernstein waves

10
Non-linear Effects

Till now we have limited our discussion exclusively to *linear* phenomena, as we used the process of linearisation. As long as the wave amplitude is small enough, the linear equations are valid. In many experiments, waves can no longer be described through a linear theory by the time they are observed. For example, according to linear theory, the amplitude of drift waves increases exponentially and the drift waves become unstable.

A wave can undergo a number of changes when its amplitude gets large. It may change its shape, say from a sine wave to lopsided triangular waveform. This is equivalent to say that Fourier components at other frequencies (or wave numbers) are generated. The nonlinear phenomena can be grouped into three broad categories:

(i) Basically non-linearizable problems

(ii) Wave-particle interactions

(iii) Wave-wave interactions

Although the problems remain to solve in the linear theory of waves and instabilities, the research in plasma is being carried out in the area of nonlinear phenomena. In this chapter we shall discuss about the nonlinear effects.

10.1 Sheaths

10.1.1 Necessity for sheaths

In the laboratories, plasma is contained in a vacuum chamber of finite size. Walls of the chamber are generally cold relative to the plasma. Let us see what happens to the plasma at the walls. For simplicity,

we consider one-dimensional case with no magnetic field (Figure 10.1). In equilibrium, the plasma has no appreciable electric field and we can take potential ϕ to be zero in the plasma. When an electron or a positive ion hits a wall of the container, it recombines there with the opposite charge and is lost. The electrons having much higher thermal velocities than those of the ions, hit the wall faster and leave the plasma with a net positive charge. Consequently, the potential of the plasma must be higher than that of the wall. That is, the potential ϕ_w of the wall is negative. Because of Debye shielding, the potential variation near the wall is within a layer of the order of several Debye lengths in thickness. Such a layer must exist on all cold walls of the chamber containing the plasma. This layer near a wall of the container is known as a *sheath*.

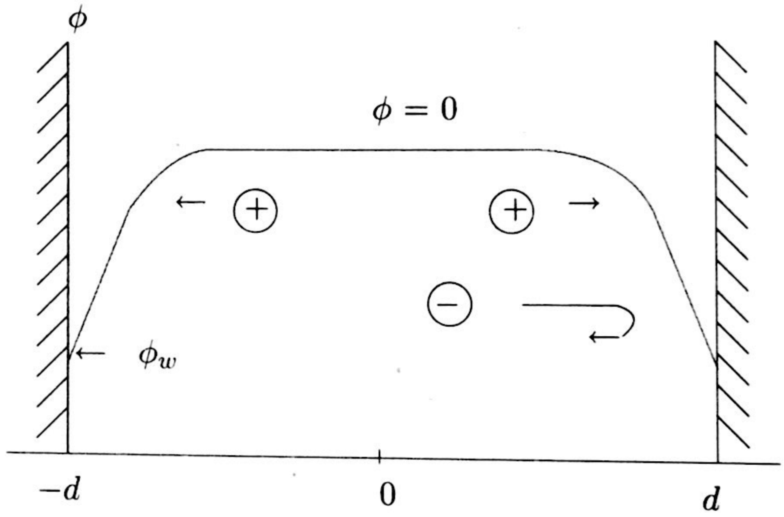

Figure 10.1: The plasma potential ϕ forms sheaths near the walls and electrons are reflected. The Coulomb barrier $e\phi_w$ adjusts itself in such a way that equal numbers of ions and electrons reach the walls per second.

The function of a sheath is obviously to form a potential barrier so that the electrons are confined in the plasma electrostatically. Now, the ions feel encouraged to go to the wall. On recombination of an ion at the wall, the potential $|\phi_w|$ decreases and the situation favours the motion of electrons. Finally, an equilibrium reaches and the Coulomb barrier $e\phi_w$ adjusts itself so that equal number of electrons and ions reach the wall per second.

10.1.2 Planar sheath equation

Let us try to understand behaviour of $\phi(x)$ in the sheath. Figure 10.2 shows situation near one of the walls of the chamber. Let at $x = 0$, ions enter the sheath region from the main plasma with a drift velocity u_0. For simplicity, let us assume $T_i = 0$ so the all the ions have the velocity u_0 at $x = 0$. Let the sheath region is collision-less and in the steady state. Suppose the potential $\phi(x)$ decreases monotonically with x. If $u(x)$ is the ion velocity, then the conservation of energy gives

$$\frac{1}{2} m_i u_0^2 + 0 = \frac{1}{2} m_i u(x)^2 + e\phi(x) \qquad u(x) = \left(u_0^2 - \frac{2e\phi(x)}{m_i}\right)^{1/2} \tag{10.1}$$

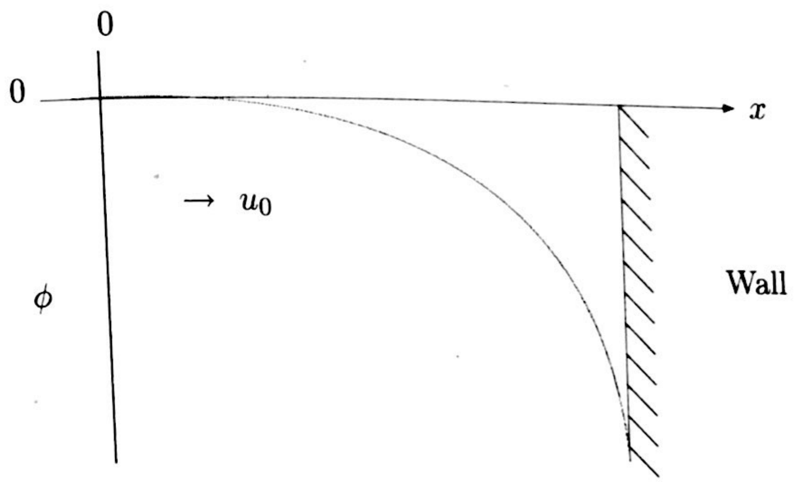

Figure 10.2: Potential ϕ in a planar sheath. Cold ions enter the sheath with a uniform velocity u_0 at $x = 0$.

If n_0 is density of ions in the main plasma, in the steady state, the equation of continuity gives[1]

$$n_0 u_0 = n_i(x) u(x) \qquad n_i(x) = \frac{n_0 u_0}{u(x)} \tag{10.2}$$

Using equation (10.1) in (10.2), we have

$$n_i(x) = n_0 u_0 \left[u_0^2 - \frac{2e\phi(x)}{m_i}\right]^{-1/2} = n_0 \left[1 - \frac{2e\phi(x)}{m_i u_0^2}\right]^{-1/2} \tag{10.3}$$

[1] The equation of continuity

$$\frac{\partial n}{\partial t} + \nabla . \,(n\vec{v}) = 0$$

reduces to $\nabla . \,(n\vec{v}) = 0$ in the steady state. That is, $n\vec{v}$ = constant.

In steady state, the electrons follow the Boltzmann relation

$$n_e(x) = n_0 \exp\Big[\frac{e\phi(x)}{KT_e}\Big] \tag{10.4}$$

The Poisson equation is

$$\epsilon_0 \nabla . \vec{E} = e(n_i - n_e)$$

For $\vec{E} = -\nabla\phi$, we have for one-dimensional case

$$-\epsilon_0 \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = e(n_i - n_e) \tag{10.5}$$

Using equations (10.3) and (10.4) in (10.5), we get

$$-\epsilon_0 \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = en_0\Big[\Big(1 - \frac{2e\phi}{m_i u_0^2}\Big)^{-1/2} - \exp\Big(\frac{e\phi}{KT_e}\Big)\Big] \tag{10.6}$$

Let us define new variables

$$u \equiv -\frac{e\phi}{KT_e} \qquad z \equiv \frac{x}{\lambda_D} = x\Big(\frac{n_0 e^2}{\epsilon_0 KT_e}\Big)^{1/2} \qquad m' \equiv \frac{u_0}{(KT_e/m_i)^{1/2}}$$

so that

$$\phi = -\frac{KT_e}{e}u \qquad x = \Big(\frac{\epsilon_0 KT_e}{n_0 e^2}\Big)^{1/2} z$$

From these relations, we have

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = -\frac{n_0 e}{\epsilon_0}\frac{\mathrm{d}^2 u}{\mathrm{d}z^2}$$

Equation (10.6) gives

$$\epsilon_0 \frac{n_0 e}{\epsilon_0}\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = en_0\Big[\Big(1 + \frac{2KT_e u}{m_i u_0^2}\Big)^{-1/2} - \exp(-u)\Big]$$

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = \Big(1 + \frac{2u}{m'^2}\Big)^{-1/2} - \mathrm{e}^{-u} \tag{10.7}$$

This is non-linear equation for a plane sheath.

10.1.3 Bohm sheath criterion

Equation (10.7) can be integrated by multiplying both sides by du/dz

$$\int_0^z \frac{\mathrm{d}^2u}{\mathrm{d}z^2}\frac{\mathrm{d}u}{\mathrm{d}z}\,\mathrm{d}z = \int_0^z \left(1+\frac{2u}{m'^2}\right)^{-1/2}\frac{\mathrm{d}u}{\mathrm{d}z}\,\mathrm{d}z - \int_0^z \mathrm{e}^{-u}\,\frac{\mathrm{d}u}{\mathrm{d}z}\,\mathrm{d}z$$

$$\int_{z=0}^{z} \frac{1}{2}\mathrm{d}(\mathrm{d}u/\mathrm{d}z)^2 = \int_0^u \left(1+\frac{2u}{m'^2}\right)^{-1/2}\mathrm{d}u - \int_0^u \mathrm{e}^{-u}\mathrm{d}u$$

$$\frac{1}{2}\left[\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2 - \left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2_{z=0}\right] = m'^2\left[\left(1+\frac{2u}{m'^2}\right)^{1/2} - 1\right] + \mathrm{e}^{-u} - 1$$

where we have used $u = 0$ at $z = 0$. When $\vec{E} = 0$ in the plasma, we have $(\mathrm{d}\phi/\mathrm{d}x)$, *i.e.*, $(\mathrm{d}u/\mathrm{d}z)$ equal to zero at $z = 0$ and therefore

$$\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}z}\right)^2 = m'^2\left[\left(1+\frac{2u}{m'^2}\right)^{1/2} - 1\right] + \mathrm{e}^{-u} - 1 \tag{10.8}$$

Integration of equation (10.8) can be done numerically. The right side of equation (10.8) must be positive for all values of u. Thus, we have

$$\left\{m'^2\left[\left(1+\frac{2u}{m'^2}\right)^{1/2} - 1\right] + \mathrm{e}^{-u} - 1\rangle\right\}\rangle\, 0$$

In particular, for $u << 1$, we have

$$\left\{m'^2\left[1+\frac{u}{m'^2} - \frac{1}{2}\,\frac{u^2}{m'^4} + \ldots - 1\right] + 1 - u + \frac{1}{2}\,u^2 + \ldots - 1\,\rangle\right\}\rangle\, 0$$

$$\left\{u - \frac{1}{2}\,\frac{u^2}{m'^2} - u + \frac{1}{2}\,u^2\,\rangle\right\}\rangle\, 0 \qquad\qquad \frac{u^2}{2}\left[-\frac{1}{m'^2}+1\right]\rangle\,\rangle\, 0$$

$$m'^2 > 1 \qquad \text{or} \qquad u_0 > (KT_e/m_i)^{1/2} \tag{10.9}$$

This inequality is known as the *Bohm sheath criterion*. It shows that ions must enter the sheath region with a velocity greater than the acoustic velocity v_s.

10.1.4 Child-Langmuir law

As the potential ϕ is negative, the $n_e(z)$ falls exponentially. Hence, in the region of large z (*i.e.*, close to the wall), the electron density can be neglected and u is large. Consequently, equation (10.7) gives

$$\frac{\mathrm{d}^2u}{\mathrm{d}z^2} \approx \left(1+\frac{2u}{m'^2}\right)^{-1/2} \approx \frac{m'}{(2u)^{1/2}} \tag{10.10}$$

Multiplying both sides of equation (10.10) by du/dz and integrating over the limit from z_s to z (where z_s is the place from where we start neglecting electron density), we get

$$\int_{z_s}^{z} \frac{d^2u}{dz^2}\frac{du}{dz}\,dz = \int_{z_s}^{z} \frac{m'}{(2u)^{1/2}}\frac{du}{dz}\,dz$$

$$\int_{z=z_s}^{z} \frac{1}{2}\,d(du/dz)^2 = \int_{u_s}^{u} \frac{m'}{(2u)^{1/2}}\,du$$

$$\frac{1}{2}\left[\left(\frac{du}{dz}\right)^2 - \left(\frac{du}{dz}\right)^2_{z=z_s}\right] = m'\sqrt{2}\left[u^{1/2}\right]_{u_s}^{u}$$

$$\left(\frac{du}{dz}\right)^2 - \left(\frac{du}{dz}\right)^2_{z=z_s} = m'2\sqrt{2}\left[u^{1/2} - u_s^{1/2}\right]$$

We can redefine the zero of u so that $u_s = 0$ at $z = z_s$. We can also neglect du/dz at $z = z_s$ as the slope of the potential curve is expected to be much steeper in the electron-free region than at the place where neglect of electron density starts. Then we have

$$\left(\frac{du}{dz}\right)^2 = m'2\sqrt{2}u^{1/2} \qquad\qquad \frac{du}{u^{1/4}} = 2^{3/4}m'^{1/2}dz$$

Integrating over z from z_s to $z_s + (d/\lambda_D) = z_w$, we get

$$\left[\frac{4}{3}u^{3/4}\right]_{z=z_s}^{z_w} = 2^{3/4}m'^{1/2}\left[z\right]_{z=z_s}^{z_w} \qquad\qquad \frac{4}{3}\left[u_w^{3/4} - u_s^{3/4}\right] = 2^{3/4}m'^{1/2}\frac{d}{\lambda_D}$$

$$\frac{4}{3}u_w^{3/4} = 2^{3/4}m'^{1/2}\frac{d}{\lambda_D}; \qquad \frac{16}{9}\,u_w^{3/2} = 2\sqrt{2}m'\frac{d^2}{\lambda_D^2}; \qquad m' = \frac{4\sqrt{2}}{9}\frac{u_w^{3/2}}{d^2}\lambda_D^2$$

Substituting back the variables, we get

$$\frac{u_0}{(KT_e/m_i)^{1/2}} = \frac{4\sqrt{2}}{9}\frac{1}{d^2}\left(\frac{e|\phi_w|}{KT_e}\right)^{3/2}\frac{\epsilon_0 KT_e}{n_0e^2}$$

Expressing the ion current $J = en_0u_0$ into the wall, we have

$$J = \frac{4}{9}\left(\frac{2e}{m_i}\right)^{1/2}\frac{\epsilon_0|\phi_w|^{3/2}}{d^2} \tag{10.11}$$

This is well known the *Child-Langmuir law*. Thus, the potential variation in the plasma-wall system can be divided into three regions: (i) The electron-free region of thickness d, given by equation (10.11), nearest to the wall. (ii) The region where potential is not zero and the electron density is aopreciable. It has the scale length of the Debye length. (iii) The pre-sheath region of the main plasma.

10.1.5 Electrostatic probes

The Bohm sheath criterion (10.9) can help in estimation of the flux of ions to a negatively biased probe in a plasma. If surface area of the probe be A and the ions entering the sheath have a drift velocity $u_0 \geq (KT_e/m_i)^{1/2}$, then the ion-current collected is

$$I = n_s e A (KT_e/m_i)^{1/2} \tag{10.12}$$

where n_s is the plasma density at the edge of the sheath. If the probe is sufficiently negative (several times KT_e) relative to the plasma to repel all but the tail of the Maxwellian electron distribution, the electron current can be neglected. Let us consider the sheath edge to be at the place where u_0 is exactly $(KT_e/m_i)^{1/2}$. For accelerating ions to this velocity, the pre-sheath potential $|\phi|$ required is

$$e|\phi| \geq \frac{1}{2}\, m_i u_0^2 = \frac{1}{2}\, m_i \frac{KT_e}{m_i} = \frac{1}{2}\, KT_e$$

Thus, potential of the sheath edge relative to the body of the plasma is

$$\phi_s \approx -\frac{1}{2}\,\frac{KT_e}{e}$$

If the electrons follow Maxwellian distribution, we have

$$n_s = n_0 \exp(e\phi_s/KT_e) = n_0 \exp(-1/2) = 0.61 n_0$$

This number may be rounded to 1/2, and using the expression for n_s in equation (10.12) we have the saturation ion-current to a negative probe

$$I_B = \frac{1}{2}\, n_0 e A (KT_e/m_i)^{1/2}$$

This I_B is sometimes called the Bohm current. Thus, for the known temperature, the plasma density can easily be obtained.

10.2 Ion acoustic shock waves

A jet traveling faster than the sound creates a shock wave, which is basically a nonlinear phenomenon.

10.2.1 Soliton

Idealized potential profile of an ion acoustic shock wave is shown in Figure 10.3. Suppose a wave is traveling to the left with a velocity u_0. An observer in the frame moving with the wave sees a stream of plasma colliding from the left on the wave with the velocity u_0. For simplicity, let us assume $T_i = 0$ so that all ions have the same velocity u_0 and let the electrons have Maxwellian distribution. From the conservation of energy, we have

$$\frac{1}{2}\, m_i u_0^2 = \frac{1}{2}\, m_i u(x)^2 + e\phi(x); \qquad u(x) = \left(u_0^2 - \frac{2e\phi(x)}{m_i}\right)^{1/2} \tag{10.13}$$

Let n_0 be the density in the undisturbed plasma, the ion density in the shock from the equation of continuity is

$$n_0 u_0 = n_i(x) u(x)$$

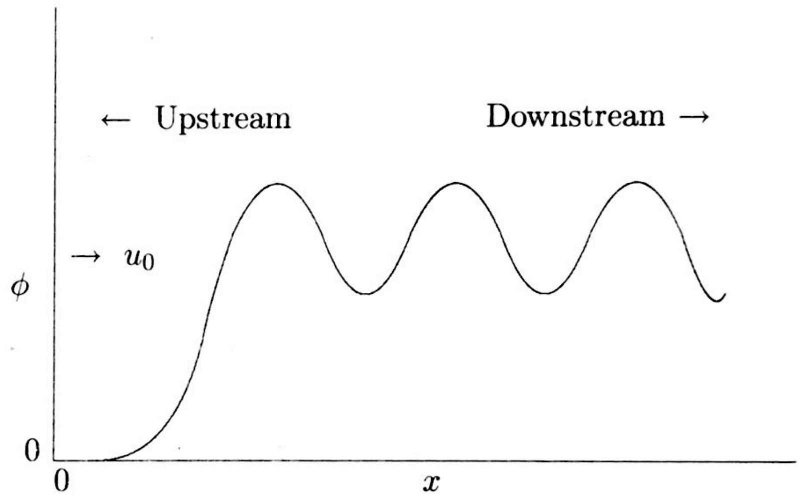

Figure 10.3: Potential distribution in an ion acoustic shock wave. The wave moves to the left, so that in the wave frame ions stream into the wave from the left with velocity u_0.

Using equation (10.13), we have

$$n_i(x) = \frac{n_0 u_0}{u(x)} = n_0 u_0 \left(u_0^2 - \frac{2e\phi}{m_i}\right)^{-1/2} = n_0 \left(1 - \frac{2e\phi}{m_i u_0^2}\right)^{-1/2} \tag{10.14}$$

In the steady state, the electrons follow the Boltzmann relation

$$n_e(x) = n_0 \exp\left(\frac{e\phi}{KT_e}\right) \tag{10.15}$$

The Poisson equation is

$$\epsilon_0 \nabla . \vec{E} = e(n_i - n_e)$$

For $\vec{E} = -\nabla\phi$, we have for one-dimensional case

$$\epsilon_0 \frac{d^2\phi}{dx^2} = e(n_e - n_i) \tag{10.16}$$

Using equations (10.14) and (10.15) in (10.16), we get

$$\epsilon_0 \frac{d^2\phi}{dx^2} = en_0\left[\exp\left(\frac{e\phi}{KT_e}\right) - \left(1 - \frac{2e\phi}{m_i u_0^2}\right)^{-1/2}\right] \tag{10.17}$$

Let us define new variables

$$u \equiv \frac{e\phi}{KT_e} \qquad z \equiv \frac{x}{\lambda_D} = x\left(\frac{n_0 e^2}{\epsilon_0 KT_e}\right)^{1/2} \qquad m' \equiv \frac{u_0}{(KT_e/m_i)^{1/2}}$$

so that

$$\phi = \frac{KT_e}{e}\, u \qquad x = \left(\frac{\epsilon_0 KT_e}{n_0 e^2}\right)^{1/2} z$$

and

$$\frac{d^2\phi}{dx^2} = \frac{n_0 e}{\epsilon_0}\frac{d^2u}{dz^2}$$

Here, m' is known as the *Mach number* of the shock, which may be defined as the velocity in the units of the acoustic velocity. Equation (10.17) gives

$$\epsilon_0 \frac{n_0 e}{\epsilon_0}\frac{d^2u}{dz^2} = en_0\left[\exp(u) - \left(1 - \frac{2KT_e u}{m_i u_0^2}\right)^{-1/2}\right]$$

$$\frac{d^2u}{dz^2} = e^u - \left(1 - \frac{2u}{m'^2}\right)^{-1/2} \tag{10.18}$$

Equation (10.18) was solved by Sagdeev by defining the right side of equation (10.18) as

$$e^u - \left(1 - \frac{2u}{m'^2}\right)^{-1/2} \equiv -\frac{dV(u)}{du} \tag{10.19}$$

so that equation (10.18) can be expressed as

$$\frac{d^2u}{dz^2} = -\frac{dV(u)}{du}$$

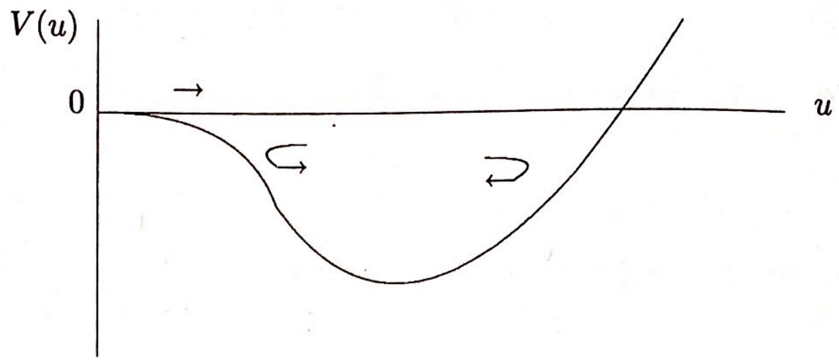

Figure 10.4: Sagdeev potential $V(u)$. The upper arrow shows the trajectory of a quasi-particle describing a soliton. It returns after reflection at the right. The lower arrows show the motion of a quasi-particle that has lost energy and is trapped in the potential well. The bouncing back and forth describes the oscillations behind a shock front.

where $V(u)$ is sometimes known as the *Sagdeev potential.* Integration of equation (10.19) with the condition $V(u) = 0$ at $u = 0$ gives

$$V(u) = 1 - \mathrm{e}^u + m'^2\left[1 - \left(1 - \frac{2u}{m'^2}\right)^{1/2}\right]$$

For the values of m' in a certain range variation of $V(u)$ versus u is shown in Figure 10.4. Sagdeev used an analogy to an oscillator in a potential well. The displacement x of an oscillator subjected to a force $-m\ \mathrm{d}V(x)/\mathrm{d}x$ is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}$$

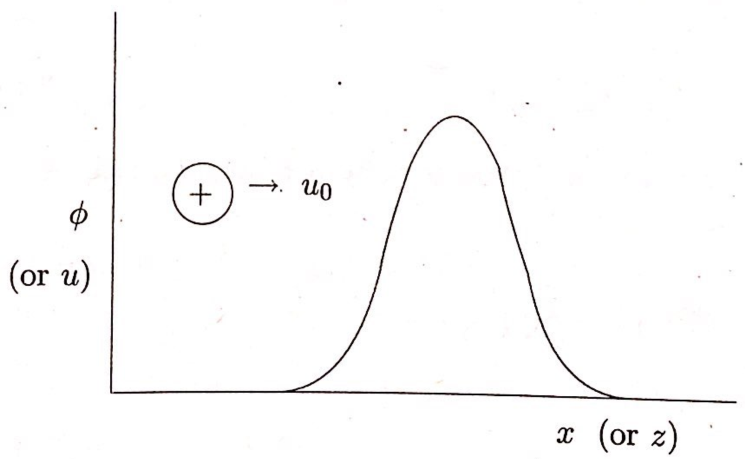

Figure 10.5: The potential in a soliton moving to left.

In this potential well, a particle entering from the left side will go to the right side of the well ($x > 0$), it is reflected back and returns to $x = 0$, making a single transit. Similarly, a quasi-particle in our analogy will make a single motion to positive u and returns to $u = 0$ as shown in Figure 10.5. Such a pulse is known as a *soliton*. It is a potential and density disturbance propagating to the left in the Figure 5 with velocity u_0.

10.2.2 Critical Mach numbers

Now, we can look into the range for m' for which the potential shown in Figure 4 is obtained. Let us first consider the fact in the region of interest, $V(u)$ is a potential well so that

$$V(u) \leq 0$$

$$1 - e^u + m'^2\left[1 - \left(1 - \frac{2u}{m'^2}\right)^{1/2}\right] \leq 0$$

For $u << 1$, we have

$$1 - \left(1 + u + \frac{1}{2}\,u^2 + \ldots\right) + m'^2\left[1 - \left(1 - \frac{u}{m'^2} - \frac{1}{2}\,\frac{u^2}{m'^4} - \ldots\right)\right] \leq 0$$

$$-u - \frac{1}{2}\,u^2 + u + \frac{1}{2}\,\frac{u^2}{m'^2} \leq 0; \qquad \frac{u^2}{2}\left[-1 + \frac{1}{m'^2}\right] \leq 0; \qquad m'^2 \geq 1$$

Thus, the lowest value of m' is 1. Let us consider fact that for some $u > 0$, the function $V(u)$ must cross u axis and there we have

$$V(u) > 0$$

$$1 - e^u + m'^2\left[1 - \left(1 - \frac{2u}{m'^2}\right)^{1/2}\right] > 0$$

$$e^u - 1 < m'^2\left[1 - \left(1 - \frac{2u}{m'^2}\right)^{1/2}\right]$$

Because of the square root, the largest value of u can be $m'^2/2$, and at this value of u, we have

$$\exp(m'^2/2) - 1 < m'^2 \qquad \text{or} \qquad m' < 1.6$$

It tells that m' is less than 1.6, giving the upper limit for m'. Thus, in a cold-ion plasma, shock waves exist only for $1 < m' < 1.6$.

10.2.3 Nonlinear Landau damping

In the linear theory, we found that for the Landau damping the decay was exponential. If the amplitude of the wave is large, it is often found that the decay is not exponential, but besides a decay in the beginning, it grows again, and then oscillates before setting down to a steady value as shown in Figure 10.6.

Trapping of a particle of velocity v occurs when its energy in the wave frame is smaller than the wave potential

$$|e\phi| > \frac{1}{2}\, m_e(v - v_\phi)^2$$

Obviousiy, small waves will trap these particles moving at high speeds near v_ϕ. In order to trap a large number of particles in the main part of the distribution (near $v = 0$) would require

$$|q\phi| = \frac{1}{2}\, m_e v_\phi^2 = \frac{1}{2}\,\frac{m_e\omega^2}{k^2} \tag{10.20}$$

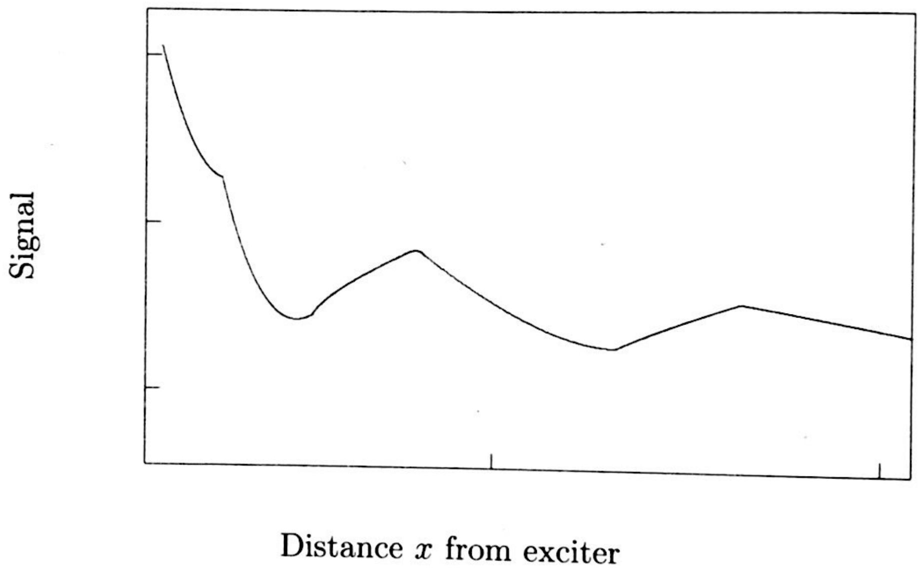

Figure 10. 6 : Typical measurement of amplitude profile of a nonlinear electron wave showing nonmonoatomic decay.

When the wave is so large, its linear behaviour can be expected to be modified greatly. Since $\vec{E} = -\nabla\phi$, we have $|\phi| = |E/k|$ and the condition (10.20) is equivalent to

$$q\,\frac{E}{k} = \frac{1}{2}\,\frac{m_e\omega^2}{k^2}$$

$$\omega = \sqrt{\frac{2kqE}{m_e}} \approx \omega_B \qquad \text{where} \qquad \omega_B^2 \equiv \left|\frac{2kqE}{m_e}\right| \qquad (10.21)$$

Here, ω_B is called the bounce-frequency because it is the frequency of oscillations of a particle trapped at the bottom of a sinusoidal potential well (Figure 10.7).

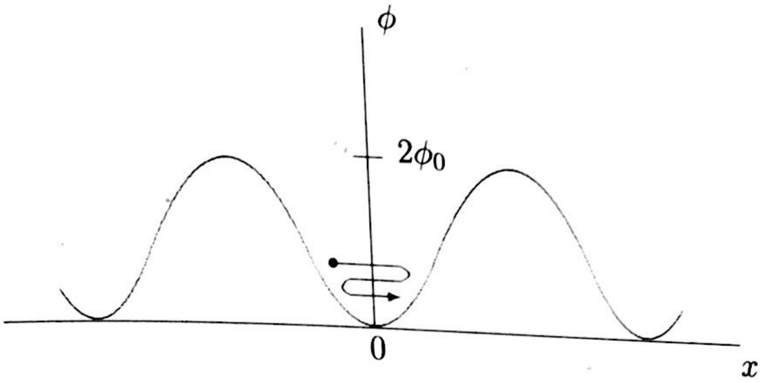

Figure 10.7: A trapped particle bounces in the potential well of a wave.

The potential is given by

$$\phi = \phi_0(1 - \cos kx) = \phi_0\ (\frac{1}{2}\ k^2x^2 + \ldots)$$

Equation of simple harmonic motion is

$$-m\omega^2 x = m\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = qE = -q\ \frac{\mathrm{d}\phi}{\mathrm{d}x} = -qk\phi_0 \sin(kx)$$

When x is small, $\sin kx \approx kx$ and we have

$$\omega = \sqrt{\frac{qk^2\phi_0}{m}}$$

and ϕ is approximately parabolic. Then ω takes the value ω_B defined by (10.21). When the resonant particles are reflected by the potential, they give kinetic energy back to the wave and the amplitude increases. When the particles bounce again from the other side, the energy goes back into the particles, and the wave is damped. Thus, one would expect oscillations in amplitude at the frequency ω_B in the wave frame. In the laboratory frame, the frequency would be $\omega' = \omega_B + kv_\phi$ and the amplitude oscillations would have wave number $k' = \omega'/v_\phi = k[1 + (\omega_B/\omega)]$.

10.3 Problems and questions

1. Describe sheaths in a plasma and obtain Bohm sheath criterion.

2. Show that for shock waves in a cold-ion plasma, the Mach number can be in between 1 and 1.6.

3. Write short notes on the following

   (i) Sheaths in a plasma

   (ii) Electrostatic probes

   (iii) Soliton

   (iv) Nonlinear Landau damping

11

Applications

After II World War, interest in plasma revived due to its possible applications in the generation of power through the process of fusion in thermonuclear reactions. In the preceding chapters, we have discussed the principles involved in the working processes of plasma. However, the actual problem, that is standing as a main hurdle in the way to achieve goal, is the problem of plasma confinement. Though a lot of work has been done on this problem, but still the problem is not fully sorted out. Further, it is not yet clear how the losses suffered by a confined plasma can be minimized due to instabilities which are generally set in. It however would be possible to derive useful power by thermonuclear fusion process.

Ionosphere plays an important role in propagation of radio waves on the surface of the earth from one place to another one. The ionosphere has been responsible for long distance radio communications. The radio signals sent from the surface of the earth are reflected back by these ionized layers (ionosphere) so that the signals are received at another place on the surface of the earth.

11.1 Propagation of radio waves through the ionosphere

Let us try to understand the transmission process of radio waves from one place on the surface of earth to another where the ionosphere is used. We are aware of the fact that the charge density in the ionosphere is not homogeneous. However, when the ionization density does not change appreciably over a distance of one wavelength, then the propagation of the radio signals can be treated in a way similar to the propagation of a ray in the geometrical optics. Let us consider an ionized medium

with electron density n. Suppose, an electromagnetic wave, expressed as $E = E_0 \sin \omega t$ is propagating through the medium. Here, E_0 is the amplitude and ω the angular frequency of the wave. Under the action of the electromagnetic field, the electrons will vibrate. Suppose m is the mass of an electron then neglecting collisions of electrons with the neutral particles, the equation of motion of the electrons is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = eE_0 \sin \omega t$$

where v is the velocity in the direction of the field. Thus, we have

$$v = -\frac{eE_0}{m\omega}\cos \omega t + C$$

where C is a constant of integration. When we consider that $v = 0$ when $\omega t = \pi/2$, then $C = 0$ and we have

$$v = -\frac{eE_0}{m\omega}\cos \omega t$$

The induction current i_c is

$$i_c = \frac{nev}{\epsilon_0} = -\frac{ne^2E_0}{\epsilon_0 m\omega}\cos \omega t$$

The displacement current i_c is

$$i_d = \frac{\mathrm{d}E}{\mathrm{d}t} = E_0\omega \cos \omega t$$

The total current i is therefore

$$i = i_c + i_d = E_0\omega \cos \omega t\Big(1 - \frac{ne^2}{\epsilon_0 m\omega^2}\Big)$$

$$= E_0\omega \cos \omega t\Big(1 - \frac{\omega_p^2}{\omega^2}\Big)$$

where $\omega_p = \sqrt{ne^2/m\epsilon_0}$ is known as the *plasma frequency.* It shows that the presence of free electrons reduces the dielectric constant of the ionized medium from 1 to $1 - (\omega_p^2/\omega^2)$. For the free space, the speed of light is c and the phase velocity becomes

$$u = \frac{c}{\sqrt{1 - (\omega_p^2/\omega^2)}} \tag{11.1}$$

The ionosphere is a region of conducting medium where the electron density increases gradually with height from the surface of the earth. Equation (11.1) shows that with the increase of height, the plasma frequency increases and thus the velocity of wave will gradually increase with its propagation in the upward direction. Let us consider a section of the wavefront AB incident on the ionized stratum where the electron density is gradually increasing (Figure 11.1). As the part A of the wavefront is moving in the region of higher electron density, the velocity of the part A of the wavefront is larger than that of the part B. Hence, the upper portion bends more than the lower one. towards the ground. As the wave propagates further in the region of higher electron density, the bending goes on increasing further and further, and ultimately proceeds downwards. Hence, the ionosphere surrounding the earth will send the radio waves back to the earth, which in absence of the ionosphere would have proceeded directly and never returned back to the earth.

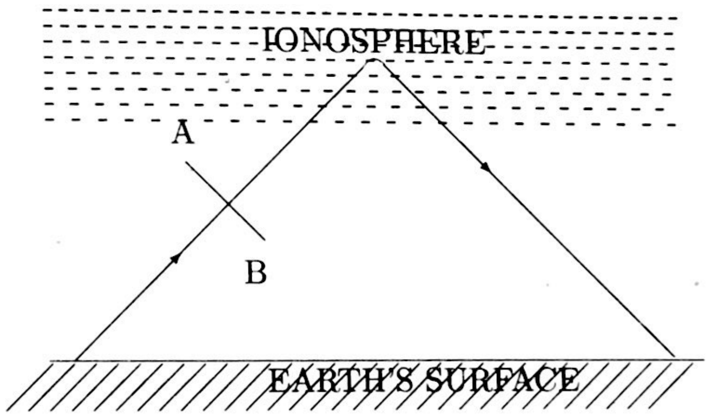

Figure 11.1: A section of the wave front AB incident on the ionized stratum where the electron density is increasing gradually with height from the earth's surface.

For finding out the condition of reflection quantitatively, we note that the refractive index of ionosphere-layer is

$$\mu = \frac{c}{u} = \sqrt{1 - (\omega_p^2/\omega^2)} \tag{11.2}$$

Let us consider Figure 11.2 showing a wave incident on the ionosphere-layer where electron density is increasing gradually upwards. Let the angle of incidence be i. Then considering that the layers of equal ionization are

horizontal and the refractive index of the region just below AB is unity, the angle of refraction is

$$\sin r = \frac{\sin i}{\mu}$$

where μ is the refractive index of the ionized layer at the point of refraction. The direction of the ray will become horizontal, *i.e.*, the wave will be totally reflected when $r = \pi/2$ and when μ represents the refractive index at the point of reflection then

$$\sin i = \mu = \sqrt{1 - (\omega_p^2/\omega^2)}$$

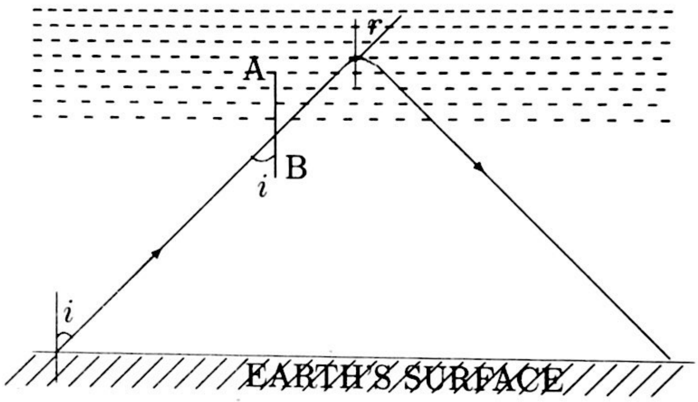

Figure 11.2: A section of wave front AB incidents on the ionized stratum where the electron density is gradually increasing in the upward direction.

When the wave is incident on the ionized layer vertically, then $i = 0$ and the condition of reflection is obtained when $\mu = 0$. Thus, we have

$$\omega_p^2 = \omega^2$$

It is obvious that after knowing the frequency of the radio wave propagating, the electron density n at the particular point at which the reflection is taking place can be determined.

11.1.1 Effect of collision on reflection of radio waves

Ionosphere is a low density plasma and the free-electron density is of the order of 10^{12} m^{-3}. That is, there is a large number of neutral particles and positive ions. Thus, when the electrons are oscillating

under the influence of the electromagnetic field of radio waves, there will be collisions of electrons with neutral particles and positive ions. Here, we would like to account for this effect. The equation of motion of electron under the action of high frequency electric field when collision of electrons with neutral particles are accounted for is

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + m\omega_c v = eE_0\, \mathrm{e}^{i\omega t} \tag{11.3}$$

where ω_c is the collision frequency and $m\omega_c v$ is the rate of change of momentum. Let us take $v = A\, \mathrm{e}^{i\omega t}$, where A is a constant. For this substitution, equation (11.3) gives

$$Am(\omega_c + i\omega) = eE_0 \qquad\qquad A = \frac{eE_0}{m(\omega_c + i\omega)}$$

so that

$$v = \frac{eE_0}{m(\omega_c + i\omega)}\, \mathrm{e}^{i\omega t}$$

and the conduction current i_c is

$$i_c = \frac{ne^2E_0}{\epsilon_0 m(\omega_c + i\omega)}\, \mathrm{e}^{i\omega t} = \frac{ne^2}{\epsilon_0 m i\omega(\omega_c + i\omega)}\frac{\mathrm{d}E}{\mathrm{d}t}$$

$$= \frac{ne^2(\omega_c - i\omega)}{\epsilon_0 m i\omega(\omega_c^2 + \omega^2)}\frac{\mathrm{d}E}{\mathrm{d}t} = \left[-\frac{ne^2}{\epsilon_0 m(\omega_c^2 + \omega^2)} - i\frac{ne^2\omega_c}{\epsilon_0 m\omega(\omega_c^2 + \omega^2)}\right]\frac{\mathrm{d}E}{\mathrm{d}t}$$

Besides the conduction current, there will be a displacement current. When ϵ is the permittivity, the displacement current i_d is

$$i_d = \frac{\mathrm{d}E}{\mathrm{d}t}$$

Thus, the total current i is

$$i = i_c + i_d = \left[-\frac{ne^2}{m\epsilon_0(\omega_c^2 + \omega^2)} - i\frac{ne^2\omega_c}{\epsilon_0 m\omega(\omega_c^2 + \omega^2)} + 1\right]\frac{\mathrm{d}E}{\mathrm{d}t} \tag{11.4}$$

When ϵ' is the effective relative permittivity of the ionosphere then we have

$$i = \epsilon'\frac{\mathrm{d}E}{\mathrm{d}t} \tag{11.5}$$

From equations (11.4) and (11.5), we have

$$\epsilon' = 1 - \frac{ne^2}{\epsilon_0 m(\omega_c^2 + \omega^2)} - i\frac{ne^2\omega_c}{\epsilon_0 m\omega(\omega_c^2 + \omega^2)}$$

The real part σ of conductivity is

$$\sigma = \frac{ne^2\omega_c}{\epsilon_0 m(\omega_c^2 + \omega^2)} = \frac{\omega_p^2\omega_c}{\omega_c^2 + \omega^2}$$

and thus, we have

$$\epsilon' = 1 - \frac{\sigma}{\omega_c} - i\,\frac{\sigma}{\omega} \tag{11.6}$$

Equation (11.6) therefore gives an expression for dielectric constant of a partially conducting medium. Thus, we can write

$$E = E_0\ \mathrm{e}^{i\omega(t-\sqrt{\epsilon'}x/c)} \tag{11.7}$$

Let us put $\sqrt{\epsilon'} = \mu - i\chi$, where the real part μ is refractive index and the imaginary part χ has a significance that will be discussed after a while. From equation (11.7), we have

$$E = E_0\ \mathrm{e}^{-\omega\chi x/c}\ \mathrm{e}^{i\omega(t-\mu x/c)} \tag{11.8}$$

The term $(\omega\chi x/c)$ thus represents attenuation of the wave and χ is the absorption per length of path c/ω $(= \lambda/2\pi)$. Suppose R is the absorption per unit length, we have $\chi = cR/\omega$. Using $\epsilon' = (\mu - i\chi)^2$ in equation (11.6), we get

$$(\mu - i\chi)^2 = 1 - \frac{\sigma}{\omega_c} - i\,\frac{\sigma}{\omega}$$

$$\mu^2 - \chi^2 - i2\mu\chi = 1 - \frac{\sigma}{\omega_c} - i\,\frac{\sigma}{\omega}$$

Therefore, we have

$$\mu^2 - \chi^2 = 1 - \frac{\sigma}{\omega_c} \qquad \text{and} \qquad \mu\chi = \frac{\sigma}{2\omega}$$

In a usual case, we have $\chi << \mu$ and therefore

$$\mu^2 = 1 - \frac{\sigma}{\omega_c} = 1 - \frac{\omega_p^2}{\omega_c^2 + \omega^2} \qquad \text{or} \qquad \mu = \sqrt{1 - \frac{\omega_p^2}{\omega_c^2 + \omega^2}} \tag{11.9}$$

Comparison of equation (11.9) with (11.2) shows the effect of the collisions. This expression however holds only for a limited range. Detailed calculations shows that the true value of μ begins to change considerably from that given by this simple expression, as $\omega_p^2/(\omega_c^2 + \omega^2)$ tends to unity for small values of ω.

11.2 Magnetohydrodynamic generator

The foundation of generation of electric power is based on Faraday's law of electromagnetic induction. This law says that when a conductor is moved across a magnetic field, the electromotive force developed across its two ends is proportional to $\mathrm{d}\Phi/\mathrm{d}t$, where Φ is the flux per unit area in the magnetic field. That is, for crossing the flux, we have to do work. In this process, the mechanical energy of rotor is converted into the electrical energy. In a generator, instead of moving conductor in a translational direction, the conductor is rotated within the pole pieces of the magnet. In a hydroelectric generator, the energy required for rotation of the conductor is provided by the potential energy of the river water. In a turbine, the rotation is made by the high speed flow of steam or fossil fuel.

Instead of a solid conductor, when a conducting fluid (gas or liquid) is allowed to flow through the magnetic field, the system is then known as a *magnetohydrodynamic generator*. Though this idea was suggested by Faraday in 1831, but conducting fluid was not available in those days. Now, with the development of plasma, a conducting fluid is available and the work on generating power by utilizing an ionized gas as a conducting fluid is going on in several laboratories world wide.

11.2.1 Basic theory

The problem of magnetohydrodynamic power generation can be understood with the help of some known fundamental principles. A stream of partially ionized gas, consisting of electrons, positive ions and neutral particles, is directed perpendicular to the direction of a magnetic field. Now the motions of individual particles are supposed to provide information how directed kinetic energy of slightly ionized gas stream is converted into electrical energy of a magnetohydrodynamic generator. For writing equations of motion for the three species, we take time derivative in the frame moving with the particle. Suppose a particle of mass m is moving with velocity $\vec{v}$. The rate of change of momentum is

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = m\left[\frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \vec{v}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \vec{v}}{\partial z}\frac{\partial z}{\partial t}\right]$$

$$= m\left[\frac{\partial \vec{v}}{\partial t} + v_x \frac{\partial \vec{v}}{\partial x} + v_x \frac{\partial \vec{v}}{\partial x} + v_x \frac{\partial \vec{v}}{\partial x}\right]$$

$$= m\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\,.\nabla)\,\vec{v}\,\right] \tag{11.10}$$

Here, for non-relativistic motion, mass of a particle is considered constant. For writing equations of motion of three species, we accounted for the forces that arise (i) due to interaction of charged particles with electric field $\vec{E}$ and magnetic field $\vec{B}$, (ii) due to pressure gradient (transfer form one region to another) and (iii) due to transfer of momentum due to collision. If n denotes the number of particles per unit volume, p the pressure, ν the frequency of collision, the equations of motion for positive ions, electrons and particles are (here suffices i, e and a correspond respectively to ion, electron and particle)

$$n_i m_i\left[\frac{\partial \vec{v_i}}{\partial t} + (\vec{v_i}\,.\nabla)\,\vec{v_i}\,\right] = en_i\left[\,\vec{E} + \frac{\vec{v_i}\times\vec{B}}{c}\right] - \nabla p_i - n_i m_i \nu_{ie}(\vec{v_i} - \vec{v_e})$$
$$-n_i m_i \nu_{ia}(\vec{v_i} - \vec{v_a}) \tag{11.11}$$

$$n_e m_e\left[\frac{\partial \vec{v_e}}{\partial t} + (\vec{v_e}\,.\nabla)\,\vec{v_e}\,\right] = -en_e\left[\,\vec{E} + \frac{\vec{v_e}\times\vec{B}}{c}\right] - \nabla p_e - n_e m_e \nu_{ei}(\vec{v_e} - \vec{v_i})$$
$$-n_e m_e \nu_{ea}(\vec{v_e} - \vec{v_a}) \tag{11.12}$$

$$n_a m_a\left[\frac{\partial \vec{v_a}}{\partial t} + (\vec{v_a}\,.\nabla)\,\vec{v_a}\,\right] = -\nabla p_a - n_a m_a \nu_{ai}(\vec{v_a} - \vec{v_i})$$
$$-n_a m_a \nu_{ae}(\vec{v_a} - \vec{v_e}) \tag{11.13}$$

Here, ν_{xy} denotes the frequency of collisions of x specie on the y specie. The boundary conditions which hold in the duct in case of a magneto-hydrodynamic generator can be expressed as the following:

(1) There is steady state so that at a point in the duct, the velocity variation with time of each component is zero, so that $\partial/\partial t = 0$.

(2) The plasma is electrically neutral, so that $n_i = n_e = n$.

(3) Changes in the velocity components are small, so that

$$(\vec{v_i}\,.\nabla)\,\vec{v_i}\,-(\vec{v_e}\,.\nabla)\,\vec{v_e}=0$$

The mass densities of electron and ion, ρ_e and ρ_i, respectively are $\rho_e = m_e n$ and $\rho_i = m_i n$. Let us consider the following transformations:

$$\rho = \rho_e + \rho_i; \qquad \vec{v}=\frac{\rho_i\vec{v_i}\,+\rho_e\vec{v_e}}{\rho}; \qquad \vec{j}= ne(\vec{v_i}-\vec{v_e})$$

where $\vec{j}$ is the current. From these relations, we have

$$\vec{v_i} = \vec{v}\,+\frac{\rho_e}{\rho ne}\vec{j} \qquad \text{and} \qquad \vec{v_e} = \vec{v}\,-\frac{\rho_i}{\rho ne}\vec{j}$$

Multiplying equation (11.11) by m_e and (11.12) by m_i, and subtracting the latter from the former along with the assumptions (1) – (3), we get

$$e(\rho_e+\rho_i)\vec{E}\,+\frac{enm_e}{c}\left(\vec{v}\,+\frac{\rho_e}{\rho ne}\vec{j}\right)\times\vec{B}\,+\frac{enm_i}{c}\left(\vec{v}\,-\frac{\rho_i}{\rho ne}\vec{j}\right)\times\vec{B}\,+m_i\nabla p_e$$
$$= m_e\nabla p_i - nm_em_i\Big[(\nu_{ei}+\nu_{ie})(\vec{v_e}-\vec{v_i}) + \nu_{ea}(\vec{v_e}-\vec{v_a}) - \nu_{ia}(\vec{v_i}-\vec{v_a})\Big]$$

Last two terms in the square bracket can be expressed as

$$\nu_{ea}(\vec{v_e}-\vec{v_a}) - \nu_{ia}(\vec{v_i}-\vec{v_a}) = \nu_{ea}\left[\vec{v}\,-\frac{\rho_i\vec{j}}{\rho ne}-\vec{v_a}\right] - \nu_{ia}\left[\vec{v}\,+\frac{\rho_e\vec{j}}{\rho ne}-\vec{v_a}\right]$$
$$= (\nu_{ea}-\nu_{ia})\,\vec{v}\,-\frac{\rho_i\nu_{ea}+\rho_e\nu_{ia}}{\rho ne}\,\vec{j}\,-(\nu_{ea}-\nu_{ia})\,\vec{v_a}$$

Thus, we have

$$0 = e\rho\vec{E}\,+\frac{en(m_i+m_e)}{c}(\vec{v}\times\vec{B}) + \frac{m_e\rho_e - m_i\rho_i}{c\rho}(\vec{j}\times\vec{B}) + m_i\nabla p_e$$
$$-m_e\nabla p_i - \frac{m_em_i}{e}(\nu_{ei}+\nu_{ie})\,\vec{j}\,+nm_im_e\Big[(\nu_{ea}-\nu_{ia})\vec{v}$$
$$-\frac{\rho_i\nu_{ea}+\rho_e\nu_{ia}}{\rho ne}\vec{j}\,-(\nu_{ea}-\nu_{ia})\,\vec{v_a}\,\Big]$$

$$0 =\vec{E}\,+\frac{\vec{v}\times\vec{B}}{c} + \frac{(\rho_e-\rho_i)}{ne\rho c}(\vec{j}\times\vec{B}) + \frac{m_i\nabla p_e - m_e\nabla p_i}{e\rho}$$
$$-\frac{m_em_i}{e^2\rho}\left[(\nu_{ei}+\nu_{ie}) + \frac{\rho_i\nu_{ea}+\rho_e\nu_{ia}}{\rho}\right]\vec{j}\,+\frac{nm_em_i(\nu_{ea}-\nu_{ia})}{e\rho}\,\vec{v}$$
$$-\frac{nm_em_i(\nu_{ea}-\nu_{ia})\,\vec{v_a}}{e\rho}$$

$$\vec{E} + \frac{\vec{v} \times \vec{B}}{c} + \frac{m_e - m_i}{e\rho c}(\vec{j} \times \vec{B}) + \frac{m_i \nabla p_e - m_e \nabla p_i}{e\rho} + \alpha(\vec{v} - \vec{v_a}) = \frac{\vec{j}}{\sigma}$$

where a constant α and electrical conductivity σ are

$$\alpha = \frac{n m_e m_i(\nu_{ea} - \nu_{ia})}{e\rho} \qquad \frac{1}{\sigma} = \frac{m_e m_i}{e^2\rho}\left[(\nu_{ei} + \nu_{ie}) + \frac{\rho_i \nu_{ea} + \rho_e \nu_{ia}}{\rho}\right]$$

The terms on the left side are respectively the electrostatic effect, electromagnetic effect, Hall effect, concentration effect and frictional field effect.

11.2.2 Principle of working

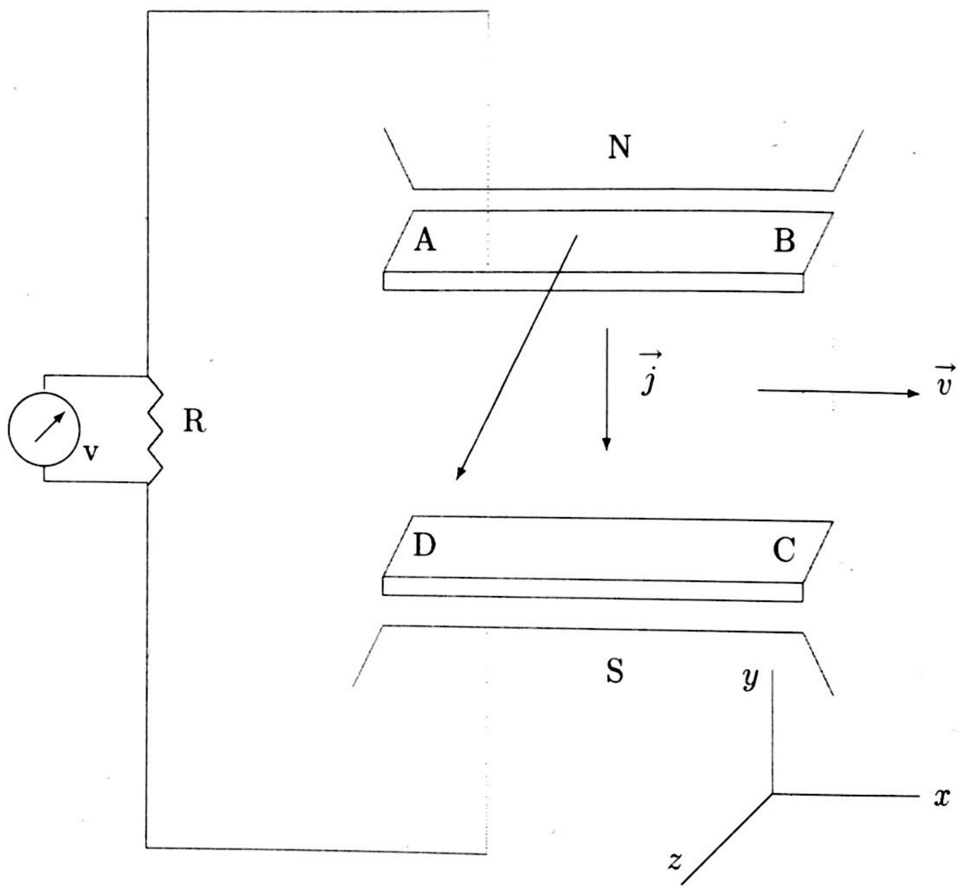

Figure 11.3: Schematic diagram of the MHD generator

Circuit diagram of simple form of an MHD generator is shown in Figure 11.3. It consists of a duct (channel) through which the working fluid, consisting of positive ions, electrons and neutral particle, is allowed to flow. The duct is placed between poles of a permanent magnet whose

electrodes are placed at the upper and lower parts of the duct. The fluid is allowed to flow in a direction perpendicular to the direction of the applied magnetic field $\vec{B}$. A charged particle moving with velocity $\vec{v}$ experiences a force in the direction perpendicular to both $\vec{v}$ and $\vec{B}$. The oppositely charged particles move in opposite directions and current is developed. This current is collected by a pair of electrodes placed on opposite of the duct in contact with the gas and connected externally through a load as shown in figure.We assume plasma to have uniform concentration and neglect frictional force and Hall effect. The magnitude of current density through a weakly ionized gas is

$$\vec{j} = \sigma\left[\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right] \tag{11.14}$$

where σ is the electrical conductivity of the charged fluid.

11.2.3 Faraday generator

In a simple case, let us neglect concentration effect, frictional term and the Hall effect. Thus, as in figure, we have $\vec{v} = v\hat{i}$, $\vec{B} = B\hat{k}$ and therefore from equation (11.14), we have

$$E_y - vB/c = j_y/\sigma$$

In an MHD generator, the charged particle gradients are small except in the immediate vicinity of channel boundaries where a sheath exists. Thus, the charged particle motion due to diffusion as well as heat conduction can be neglected. The electric field considered here results from the potential difference between the electrodes. For convenience, let us assume ν and σ to be constant. Thus, from figure, we have

$$j_y = \sigma[E_y - vB/c]$$

At open circuit, $j_y = 0$. Thus, the open circuited electric field is $E_y = vB/c$. We introduce a term $k = cE_y/vB$, called the loading parameter. Then, we have

$$j_y = -\frac{\sigma v B}{c}(1 - k)$$

The loading parameter can assume values between 0 and 1. Here, the negative sign indicates that with the geometry of the magnetic field and the direction of the fluid flow, the current flows in the negative direction of y-axis. The electric power delivered per unit volume of the MHD generator is

$$P = -j_y E_y = \frac{\sigma v^2 B^2}{c^2} k(1-k)$$

This power will be maximum when

$$\frac{dP}{dk} = 0 \qquad 1 - 2k = 0 \qquad k = \frac{1}{2}$$

We have

$$P_{max} = \frac{\sigma v^2 B^2}{4c^2}$$

The rate at which the work is done by the gas in pushing itself through the magnetic field is $j_y vB/c$. Thus, the efficiency of the MHD generator is

$$\eta = \frac{j_y E}{j_y vB/c} = k$$

11.2.4 Faraday generator when Hall effect is accounted for

Neglecting concentration and frictional terms but retaining the Hall effect terms in the generalized Ohm,s law we have

$$\vec{E} + \vec{v} \times \vec{B} - \frac{1}{\rho e}(m_i - m_e)\, \vec{j} \times \vec{B} = \frac{\vec{j}}{\sigma}$$

where the electrical conductivity σ is

$$\frac{1}{\sigma} = \frac{m_e m_i}{e^2 \rho}\left[(\nu_{ei} + \nu_{ie}) + \frac{\rho_i \nu_{ea} + \rho_e \nu_{ia}}{\rho}\right]$$

When we consider only the electron particle collisions, we have

$$\frac{1}{\sigma} = \frac{m_e m_i \rho_i \nu_{ea}}{e^2 \rho^2}$$

Since $m_e << m_i$, we have

$$\sigma = \frac{ne^2}{m_e \nu_{ea}}$$

Now,

$$\vec{j} = \sigma\Big[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\Big] + \frac{\sigma(m_e - m_i)}{e\rho c}(\vec{j}\times\vec{B})$$

Considering $m_e << m_i$ and using σ in the second part on right side of this equation, we get

$$\vec{j} = \sigma\Big[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\Big] - \frac{eB}{m_e c\nu_{ea}}(j_y\hat{x} - j_x\hat{y})$$

$$= \sigma\Big[\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\Big] - \frac{\omega_e}{c\nu_{ea}}(j_y\hat{x} - j_x\hat{y}) \qquad (11.15)$$

where $\omega_e = eB/m_e$ is electron frequency. The Hall voltage will develop along x-axis. But, for accounting for the effect of Hall current on the efficiency of Faraday generator. We have not placed any electrode along x-axis, so we shall neglect the voltage along the x-axis. However, we shall consider the effect of Hall current on the Faraday cylinder. Thus, we have

$$j_x = -\beta j_y \qquad \text{and} \qquad j_y = \sigma[E_y - vB/c] + \beta j_x$$

where $\beta = \omega_e/c\nu_{ea}$. Thus, we have

$$j_y = \sigma(E_y - vB/c) - \beta^2 j_y$$

$$j_y = \frac{1}{(1+\beta^2)}\sigma(E_y - vB/c)$$

At open circuit, $j_y = 0$. Thus, the open circuited electric field is $E_y = vB/c$. We introduce a term $k = cE_y/vB$, called the loading parameter. Then, we have

$$j_y = -\frac{\sigma v B}{c(1+\beta^2)}(1-k)$$

The loading parameter can assume values between 0 and 1. Here, the negative sign indicates that with the geometry of the magnetic field and the direction of the fluid flow, the current flows in the negative direction of y-axis. The electric power delivered per unit volume of the MHD generator is

$$P = -j_y E_y = \frac{\sigma v^2 B^2}{c^2(1+\beta^2)}k(1-k)$$

Thus, in the presence of Hall current, the power generated is reduced by a factor of $(1+\beta^2)$.

11.3 Generation of microwave utilizing high density plasma

Plasma is a very promising medium for generating and amplifying microwaves. The oscillatory property of plasma helps as a starting point for such studies. The natural electromagnetic oscillations which take place in a plasma are incoherent and do not have any practical use. Some results for production of millimeter waves have been published by scientists. There they utilized nonlinear voltage current characteristics of a mercury arc for harmonic generation of the waves just as rectifying properties of silicon crystals have been used by the scientists for generation of millimeter waves in microwave spectroscopy. Cyclotron radiation of an electron placed in a magnetic field can also be used for production of high frequency electromagnetic waves.

11.4 Problems and questions

1. Discuss about the ionospheric plasma around the earth.
2. Discuss about the basic theory of magnetohydrodynamic generator.
3. Write short notes on the following

   (i) Ionospheric plasma

   (ii) MHD generator

Appendix

A. Useful constants

c	speed of light	3×10^8 m/s
e	electron charge	1.602×10^{-19} C
m_e	electron mass	9.109×10^{-31} kg
m_p	mass of proton	1.673×10^{-27} kg
m_p/m_e		1837
k	Boltzmann's constant	1.38×10^{-23} J/K
h	Planck constant	6.626×10^{-34} Js
R	Gas constant	8.314 J/mol K
N	Avogadro number	6.022×10^{23} mol^{-1}
ϵ_0	permittivity of free space	8.854×10^{-12} F/m
μ_0	permeability of free space	$4\,\pi \times 10^{-7}$ H/m
eV	electron volt	1.602×10^{-19} J
Å	Angstrom	10^{-10} m

B. Useful vector relations

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = [\vec{A}\vec{B}\vec{C}]$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})$$

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

$$(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = [\vec{A}\vec{B}\vec{D}]\,\vec{C} - [\vec{A}\vec{B}\vec{C}]\,\vec{D} = [\vec{A}\vec{C}\vec{D}]\,\vec{B} - [\vec{B}\vec{C}\vec{D}]\,\vec{A}$$

$$\nabla\cdot(\phi\,\vec{A}) = \vec{A}\cdot\nabla\phi + \phi\nabla\cdot\vec{A}$$

$$\nabla\times(\phi\,\vec{A}) = \nabla\phi\times\vec{A} + \phi\nabla\times\vec{A}$$

$$\vec{A}\times(\nabla\times\vec{B}) = \nabla(\vec{A}\cdot\vec{B}) - (\vec{A}\cdot\nabla)\,\vec{B} - (\vec{B}\cdot\nabla)\,\vec{A} - \vec{B}\times(\nabla\times\vec{A})$$

$$\vec{B}\times(\nabla\times\vec{B}) = \tfrac{1}{2}\,\nabla B^2 - (\vec{B}\cdot\nabla)\,\vec{B}$$

$$(\vec{A}\cdot\nabla)\,\vec{A} = \nabla(\tfrac{1}{2}\,A^2) - \vec{A}\times(\nabla\times\vec{A})$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

$$\nabla\times(\vec{A}\times\vec{B}) = \vec{A}\,(\nabla\cdot\vec{B}) - \vec{B}\,(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\,\vec{A}) - (\vec{A}\cdot\nabla)\,\vec{B})$$

$$\nabla\times[(\vec{A}\cdot\nabla)\,\vec{A}] = (\vec{A}\cdot\nabla)(\nabla\times\vec{A}) + (\nabla\cdot\vec{A})(\nabla\times\vec{A}) - [(\nabla\times\vec{A})\cdot\nabla]\,\vec{A}$$

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - (\nabla\cdot\nabla)\,\vec{A}$$

$$\nabla\times\nabla\phi = 0$$

$$\nabla\cdot(\nabla\times\vec{A}) = 0$$

C. Expressions in cylindrical coordinates (r, θ, z)

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_z\hat{z}$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\nabla.\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial\theta}A_\theta + \frac{\partial}{\partial z}A_z$$

$$\nabla\times\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{z}$$

$$\nabla^2\vec{A} = (\nabla.\nabla)\vec{A}$$

$$= \left[\nabla^2 A_r - \frac{1}{r^2}\left(A_r + 2\frac{\partial A_\theta}{\partial\theta}\right)\right]\hat{r} + \left[\nabla^2 A_\theta - \frac{1}{r^2}\left(A_\theta - 2\frac{\partial A_r}{\partial\theta}\right)\right]\hat{\theta} + \nabla^2 A_z\hat{z}$$

$$\begin{aligned}(\vec{A}.\nabla)\vec{B} = {} & \left(A_r\frac{\partial B_r}{\partial r} + A_\theta\,\frac{1}{r}\frac{\partial B_r}{\partial\theta} + A_z\frac{\partial B_r}{\partial z} - \frac{1}{r}\,A_\theta B_\theta\right)\hat{r} \\ & + \left(A_r\frac{\partial B_\theta}{\partial r} + A_\theta\,\frac{1}{r}\frac{\partial B_\theta}{\partial\theta} + A_z\frac{\partial B_\theta}{\partial z} + \frac{1}{r}A_\theta B_r\right)\hat{\theta} \\ & + \left(A_r\frac{\partial B_z}{\partial r} + A_\theta\frac{1}{r}\frac{\partial B_z}{\partial\theta} + A_z\frac{\partial B_z}{\partial z}\right)\hat{z}\end{aligned}$$

D. Evaluation of some integrals

(i) Let

$$I = \int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x = 2\int_0^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x \tag{11.16}$$

Similarly, we can write

$$I = 2\int_0^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y \tag{11.17}$$

Multiplication of equations (11.16) and (11.17), we get

$$\begin{aligned}I^2 &= 4\int_0^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x \int_0^{\infty} \mathrm{e}^{-y^2}\mathrm{d}y \\ &= 4\int_{x=0}^{\infty}\int_{y=0}^{\infty} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\ \mathrm{d}y\end{aligned}$$

If we use, $x = r\cos\theta$ and $y = r\sin\theta$

we have $dx\,dy = r\,dr\,d\theta$ and $x^2 + y^2 = r^2$

Further, the limits for x, from 0 to ∞ and for y, from 0 to ∞ cover the first quadrant of the region. The same region can be covered through the limits for r, from 0 to ∞ and for θ, from 0 to $\pi/2$. Thus, we have

$$I^2 = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-r^2}\, r\,dr\,d\theta$$

$$= 4\,\frac{\pi}{2}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr = 2\pi\left[-\frac{e^{-r^2}}{2}\right]_0^{\infty} = \pi$$

Thus, we have

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi} \qquad \text{and} \qquad \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

(ii) Similarly, we have

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \int_{-\infty}^{\infty} e^{-y^2}\frac{dy}{\sqrt{a}} \qquad \left[y = \sqrt{a}\,x\right]$$

$$= \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\frac{\pi}{a}}$$

(iii) Let

$$I = \int_{-\infty}^{\infty} x^2 e^{-ax^2}dx$$

$$= \int_{-\infty}^{\infty} x\left[xe^{-ax^2}\right]dx = \left[x\frac{e^{-ax^2}}{-2a}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{e^{-ax^2}}{-2a}\,dx$$

$$= \frac{1}{2a}\int_{-\infty}^{\infty} e^{-ax^2}dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}} = \frac{\sqrt{\pi}}{2a^{3/2}}$$

(iv) Let

$$I_n = \int_0^{\infty} e^{-ax^2} x^n\,dx$$

where n is a non-negative integer.

$$\frac{\partial I_n}{\partial a} = \int_0^{\infty} e^{-ax^2}(-x^2)x^n\,dx$$

$$= -\int_0^{\infty} e^{-ax^2} x^{n+2}\,dx = -I_{n+2}$$

This recurrence relation can help in computation a number of integrals. For that we should have the values for I_0 and I_1. We have

$$I_0 = \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

and

$$I_1 = \int_0^\infty e^{-ax^2} x \, dx$$

Let $y = ax^2$ $\qquad$ $dy = 2\,ax\,dx$

Thus,

$$I_1 = \frac{1}{2a}\int_0^\infty e^{-y} dy = \frac{1}{2a}\Big[-e^{-y}\Big]_0^\infty = \frac{1}{2a}$$

Further, integrals can be obtained with the help of the recurrence relation

$$I_2 = \int_0^\infty e^{-ax^2} x^2 \, dx = -\frac{\partial I_0}{\partial a} = -\frac{\partial}{\partial a}(\sqrt{\pi/a})/2 = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$I_3 = \int_0^\infty e^{-ax^2} x^3 dx = -\frac{\partial I_1}{\partial a} = -\frac{\partial}{\partial a}1/2a = \frac{1}{2a^2}$$

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Appendices have been added to give additional information to the reader.

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